Asset Management & Sustainable Finance
Final Examination

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January 30th, 2023

Deadline: March 23th, 2023

Remark 1 ***The final examination is composed of 2 exercises. Please write entirely your answers[1]. Precise the different concepts and the different statistics you use. Define the optimization program associated to each portfolio. Provide also one Python program by exercise.***

- ***Concerning risk decomposition[2], present the results as follows:***

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1				
2				
$\vdots$				
n				
$\mathcal{R}(x)$			✓	

- ***The report is a zipped file, whose filename is <u>yourname.zip</u> if you do the project alone or <u>yourname1-yourname2.zip</u> if you do the project in groups of two.***
- ***The zipped file is composed of three files:***
  1. ***The pdf document that contains the answers to the two exercises and a cover sheet with your names;***
  2. ***The Python program of each exercise with an explicit filename, e.g. exercise1.py.***
- ***The project seems to be very long. However, once you have understood how to solve a mean-variance optimization problem with a QP solver, you can duplicate your code for many questions. For instance, Question 2.(c) is a duplication of Question 2.(b), same thing with Questions 3.(a), 3.(b) and 3.(c). Questions 2.(a)–2.(g) is also a variant of Question 2.(b) by changing a little bit the QP objective function and adding an inequality constraint.***

[1] Read carefully the questions and answer to all the elements of the questions. For instance, when I say "Find the portfolio x and compute its volatility $\sigma(x)$", you have to give the numeric values of x and $\sigma(x)$. If you only give the numeric value of $\sigma(x)$, the answer is false because I don't know what the weights of the portfolio are.

[2] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, $\mathcal{MR}_i$ is the marginal risk, $\mathcal{RC}_i$ is the nominal risk contribution, $\mathcal{RC}_i^\star$ is the relative risk contribution and $\mathcal{R}(x)$ is the risk measure of the portfolio.

1 Portfolio optimization and risk budgeting

We consider an investment universe of $n = 6$ stocks. We assume that the expected returns μ_i are equal to 5%, 5%, 6%, 4%, 7% and 3%, whereas the volatilities σ_i are equal to 20%, 22%, 25%, 18%, 35% and 10%. The correlation matrix is equal to:

$$\mathbb{C} = (\rho_{i,j}) = \begin{pmatrix} 100\% & & & & & \\ 50\% & 100\% & & & & \\ 30\% & 30\% & 100\% & & & \\ 60\% & 60\% & 60\% & 100\% & & \\ 40\% & 30\% & 70\% & 30\% & 100\% & \\ 20\% & 20\% & 15\% & 25\% & 15\% & 100\% \end{pmatrix}$$

The risk free return is set to 1%.

1. (a) Compute the covariance matrix Σ of stock returns.
   (b) Compte the Sharpe ratio of each asset.

2. We consider **long/short MVO portfolios** such that $\sum_{i=1}^{n} x_i = 1$.

   (a) Give the QP formulation of the mean-variance optimization problem:

   $$\begin{array}{rcl} x^{\star} & = & \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \mu \\ & \text{s.t.} & \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ -10 \leq x_i \leq 10 \end{array} \right. \end{array}$$

   (b) Using the γ-problem, find the optimal solution[3] $x^{\star}(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^{\star}(\gamma))$, its volatility $\sigma(x^{\star}(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^{\star}(\gamma) \mid r)$.

   (c) Draw the efficient frontier by considering granular values[4] of γ.

   (d) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 10% and 12%. Give the corresponding values γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^{\star}(\gamma))$, its volatility $\sigma(x^{\star}(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^{\star}(\gamma) \mid r)$.

   (e) Find the tangency portfolio[5]. Compute its expected return $\mu(x^{\star}(\gamma))$, its volatility $\sigma(x^{\star}(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^{\star}(\gamma) \mid r)$.

3. We consider **long-only MVO portfolios** such that $\sum_{i=1}^{n} x_i = 1$ and $0 \leq x_i \leq 1$.

   (a) Using the γ-problem, find the optimal solution[6] $x^{\star}(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^{\star}(\gamma))$, its volatility $\sigma(x^{\star}(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^{\star}(\gamma) \mid r)$.

   (b) Compare the efficient frontier by considering granular values of γ with the long/short efficient frontier obtained in Question 2.(c). Comment on these results.

   (c) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 10% and 12%. Give the corresponding values of γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^{\star}(\gamma))$, its volatility $\sigma(x^{\star}(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^{\star}(\gamma) \mid r)$.

[3] You have to give the composition of each optimized portfolios.
[4] For instance, you can consider that $\gamma = -0.5, -0.4, \ldots, -0.1, 0, 0.05, 0.10, \ldots, 0.95, 1, 2, \ldots, 10$.
[5] You can compute it using the analytical solution or using the numerical brute force method.
[6] You have to give the composition of each optimized portfolios.

(d) Find the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Compute its expected return $\mu\left(x^{\star}_{\mathrm{MSR}}\right)$, its volatility $\sigma\left(x^{\star}_{\mathrm{MSR}}\right)$ and its Sharpe ratio $\mathrm{SR}\left(x^{\star}_{\mathrm{MSR}} \mid r\right)$.

(e) Compute the beta coefficient β_i of each asset with respect to the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Deduce the implied expected return μ_i that is priced in by the market[7], and the corresponding alpha coefficient α_i of each asset.

4. We consider risk-budgeting portfolios.

(a) Give the risk decomposition of the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$ (MSR).

(b) Give the risk decomposition of the equally-weighted portfolio (EW).

(c) Give the risk decomposition of the long-only minimum variance portfolio (MV).

(d) Give the risk decomposition of the long-only most diversified portfolio (MDP).

(e) Give the risk decomposition of the equal risk contribution portfolio (ERC).

(f) Compute the beta $\beta\left(x \mid b\right)$ of the portfolios MSR, EW, MV, MDP and ERC with respect to the benchmark b when b is the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Same question when b is the EW portfolio. Comment on these results.

5. We consider again Question 2, but we now impose a leverage constraint:

$$\mathcal{L}\left(x\right) = \sum_{i=1}^{n} \left|x_i\right| \leq \mathcal{L}^{+}$$

where $\mathcal{L}^{+}$ is the maximum leverage. The mean-variance optimization problem is then:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top}\Sigma x - \gamma x^{\top}\mu \\ &\text{s.t.} \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ \sum_{i=1}^{n} \left|x_i\right| \leq \mathcal{L}^{+} \\ -10 \leq x_i \leq 10 \end{array} \right. \end{aligned} \tag{1}$$

(a) We consider the following parameterization:

$$x_i = x_i^{+} - x_i^{-}$$

where $x_i^{+} \geq 0$ and $x_i^{-} \geq 0$ are respectively the positive and negative parts of the weight of asset i. The matrix form is then:

$$\left\{ \begin{array}{l} x = x^{+} - x^{-} \\ x^{+} \geq \mathbf{0}_n \\ x^{-} \geq \mathbf{0}_n \end{array} \right.$$

i. What is the expression of the leverage ratio $\mathcal{L}\left(x\right) = \sum_{i=1}^{n} \left|x_i\right|$ with respect to x^{+} and x^{-}.

ii. What is the expression of the expected return $\mu\left(x\right) = x^{\top}\mu$ with respect to x^{+} and x^{-}.

iii. What is the expression of the variance $\sigma^2\left(x\right) = x^{\top}\Sigma x$ with respect to x^{+} and x^{-}.

iv. We consider the canonical quadratic programming problem:

$$\begin{aligned} X^{\star} &= \arg\min \frac{1}{2} X^{\top} Q X - X^{\top} R \\ &\text{s.t.} \left\{ \begin{array}{l} AX = B \\ CX \leq D \\ X^{-} \leq X \leq X^{+} \end{array} \right. \end{aligned} \tag{2}$$

[7] We assume that the market portfolio is the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$.

By considering the following definition:

$$X = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$$

Cast the MVO problem (1) with the leverage constraint into an augmented QP problem (2), meaning that you must give the corresponding formulation of the matrices Q, R, A, B, C, D, X^- and X^+.

(b) We consider that the coefficient γ is respectively equal to 0, 0.10, 0.20, 0.50 and 1.00. Solve the augmented QP problem by considering that the maximum leverage value is equal to $\mathcal{L}^+ = 10$ and give the optimal portfolio for each value of γ. You must obtain the same solutions as those found in Question 2.(b). Why do we obtain the long/short optimized portfolios?

(c) Same question when the maximum leverage value is set to $\mathcal{L}^+ = 1$. In this case, you must obtain the same solutions as those found in Question 3.(a). Why do we obtain the optimized long-only portfolios?

(d) Find the optimal portfolios when the coefficient γ is respectively equal to 0, 0.10, 0.20, 0.50 and 1.00 and the maximum leverage value is set to $\mathcal{L}^+ = 1.5$.

2 Equity and Bond Portfolio Optimization with Green Preferences

We consider an investment universe of 8 issuers. In the table below, we report the carbon emissions $\mathcal{CE}_{i,j}$ of these companies and their revenues Y_i, and we indicate in the last row whether the company belongs to sector $\mathcal{S}_1$ or $\mathcal{S}_2$:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{CE}_{i,1}$ (in $ktCO_2e$)	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$ (in $ktCO_2e$)	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$ (in $ktCO_2e$)	24 000	15 000	1 210	550	500	187	30 000	199
Y_i (in \$ bn)	300	328	125	100	200	102	107	25
Sector	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_1$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_2$

The benchmark b of this investment universe is defined as:

$$b = (22\%, 19\%, 17\%, 13\%, 11\%, 8\%, 6\%, 4\%)$$

In what follows, we consider **long-only portfolios**.

1. We want to compute the carbon intensity of the benchmark.

   (a) Compute the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes 1, 2 and 3.

   (b) Deduce the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes $1+2$ and $1+2+3$.

   (c) Deduce the weighted average carbone intensity (WACI) of the benchmark if we consider the scopes $1+2+3$.

   (d) We assume that the market capitalization of the benchmark portfolio is equal to \$10 tn and we invest \$1 bn.

      i. Deduce the market capitalization of each company (expressed in \$ bn).

      ii. Compute the ownership ratio for each asset (expressed in bps).

      iii. Compute the carbon emissions of the benchmark portfolio[8] if we invest \$1 bn and we consider the scopes $1+2+3$.

      iv. Compare the (exact) carbon intensity of the benchmark portfolio with the WACI value obtained in Question 1.c.

2. We would like to manage **an equity portfolio** with respect to the previous investment universe and reduce the **weighted average carbon intensity** of the benchmark by the rate $\mathcal{R}$. We assume that the volatility of the stocks is respectively equal to 22%, 20%, 25%, 18%, 40%, 23%, 13% and 29%. The correlation matrix between these stocks is given by:

$$\rho = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 60\% & 100\% \end{pmatrix}$$

   (a) Compute the covariance matrix Σ.

[8] We assume that the float percentage is equal to 100% for all the 8 companies.

(b) Write the optimization problem if the objective function is to minimize the tracking error risk under the constraint of carbon intensity.

(c) Give the QP formulation of the optimization problem.

(d) $\boldsymbol{\mathcal{R}}$ is equal to 20%. Find the optimal portfolio if we target scopes $1+2$. What is the value of the tracking error volatility?

(e) Same question if $\boldsymbol{\mathcal{R}}$ is equal to 30%, 50%, and 70%.

(f) We target scopes $1+2+3$. Find the optimal portfolio if $\boldsymbol{\mathcal{R}}$ is equal to $20\%, 30\%$, 50% and 70%. Give the value of the tracking error volatility for each optimized portfolio.

(g) Comment on the results obtained in Questions 2.(d), 2.(e) and 2.(f).

3. We would like to manage **a bond portfolio** with respect to the previous investment universe and reduce the **weighted average carbon intensity** of the benchmark by the rate $\boldsymbol{\mathcal{R}}$ (scopes 1+2+3). In the table below, we report the modified duration MD_i and the duration times spread DTS_i of each corporate bond i:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8
MD_i (in year)	3.56	7.48	6.54	10.23	2.40	2.30	9.12	7.96
DTS_i (in bps)	103	155	75	796	89	45	320	245
Sector	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_1$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_2$

We define the MD and DTS metrics of portfolio x as follows:

$$\mathrm{MD}(x) = \sum_{i=1}^{n} x_i \cdot \mathrm{MD}_i$$

and:

$$\mathrm{DTS}(x) = \sum_{i=1}^{n} x_i \cdot \mathrm{DTS}_i$$

The tracking error risk (or active risk) can be calculated using three functions. For the active share risk, we have:

$$\sigma_{\mathrm{AS}}(x \mid b) = \sqrt{\sum_{i=1}^{n} (x_i - b_i)^2}$$

We also consider the MD-based tracking error risk:

$$\sigma_{\mathrm{MD}}(x \mid b) = \sqrt{\left(\sum_{i \in \mathcal{S}_1} (x_i - b_i)\,\mathrm{MD}_i\right)^2 + \left(\sum_{i \in \mathcal{S}_2} (x_i - b_i)\,\mathrm{MD}_i\right)^2}$$

and the DTS-based tracking error risk:

$$\sigma_{\mathrm{DTS}}(x \mid b) = \sqrt{\left(\sum_{i \in \mathcal{S}_1} (x_i - b_i)\,\mathrm{DTS}_i\right)^2 + \left(\sum_{i \in \mathcal{S}_2} (x_i - b_i)\,\mathrm{DTS}_i\right)^2}$$

We note $\mathcal{R}_{\mathrm{AS}}(x \mid b) = \frac{1}{2}\sigma^2_{\mathrm{AS}}(x \mid b)$, $\mathcal{R}_{\mathrm{MD}}(x \mid b) = \frac{1}{2}\sigma^2_{\mathrm{MD}}(x \mid b)$ and $\mathcal{R}_{\mathrm{DTS}}(x \mid b) = \frac{1}{2}\sigma^2_{\mathrm{MD}}(x \mid b)$. Finally, we define a synthetic risk measure as a combination of AS, MD and DTS active risks:

$$\mathcal{R}(x \mid b) = \varphi_{\mathrm{AS}}\mathcal{R}_{\mathrm{AS}}(x \mid b) + \varphi_{\mathrm{MD}}\mathcal{R}_{\mathrm{MD}}(x \mid b) + \varphi_{\mathrm{DTS}}\mathcal{R}_{\mathrm{DTS}}(x \mid b)$$

where $\varphi_{\mathrm{AS}} \geq 0$, $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the weight of each risk. In what follows, we use the following numerical values: $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 1$. The reduction rate $\boldsymbol{\mathcal{R}}$ of the weighted average carbon intensity is set to 50% for scopes 1, 2 and 3.

(a) Compute the modified duration $\mathrm{MD}(b)$ and the duration times spread $\mathrm{DTS}(b)$ of the benchmark.

(b) Let x be the equally-weighted portfolio. Compute[9] $\mathrm{MD}(x)$, $\mathrm{DTS}(x)$, $\sigma_{\mathrm{AS}}(x \mid b)$, $\sigma_{\mathrm{MD}}(x \mid b)$ and $\sigma_{\mathrm{DTS}}(x \mid b)$.

(c) Solve the following optimization problem[10]:

$$\begin{aligned} x^{\star} &= \arg\min \mathcal{R}_{\mathrm{AS}}(x \mid b) \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ \mathrm{MD}(x) = \mathrm{MD}(b) \\ \mathrm{DTS}(x) = \mathrm{DTS}(b) \\ \boldsymbol{\mathcal{CI}}(x) \leq (1 - \boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{CI}}(b) \\ 0 \leq x_i \leq 1 \end{array} \right. \end{aligned}$$

Compute $\mathrm{MD}(x^{\star})$, $\mathrm{DTS}(x^{\star})$, $\sigma_{\mathrm{AS}}(x^{\star} \mid b)$, $\sigma_{\mathrm{MD}}(x^{\star} \mid b)$ and $\sigma_{\mathrm{DTS}}(x^{\star} \mid b)$.

(d) Solve the following optimization problem:

$$\begin{aligned} x^{\star} &= \arg\min \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}(x \mid b) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}(x \mid b) \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ \mathrm{DTS}(x) = \mathrm{DTS}(b) \\ \boldsymbol{\mathcal{CI}}(x) \leq (1 - \boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{CI}}(b) \\ 0 \leq x_i \leq 1 \end{array} \right. \end{aligned}$$

Compute $\mathrm{MD}(x^{\star})$, $\mathrm{DTS}(x^{\star})$, $\sigma_{\mathrm{AS}}(x^{\star} \mid b)$, $\sigma_{\mathrm{MD}}(x^{\star} \mid b)$ and $\sigma_{\mathrm{DTS}}(x^{\star} \mid b)$.

(e) Solve the following optimization problem:

$$\begin{aligned} x^{\star} &= \arg\min \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}(x \mid b) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}(x \mid b) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}(x \mid b) \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ \boldsymbol{\mathcal{CI}}(x) \leq (1 - \boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{CI}}(b) \\ 0 \leq x_i \leq 1 \end{array} \right. \end{aligned}$$

Compute $\mathrm{MD}(x^{\star})$, $\mathrm{DTS}(x^{\star})$, $\sigma_{\mathrm{AS}}(x^{\star} \mid b)$, $\sigma_{\mathrm{MD}}(x^{\star} \mid b)$ and $\sigma_{\mathrm{DTS}}(x^{\star} \mid b)$.

(f) Comment on the results obtained in Question 3.(c), 3.(d) and 3.(e).

Remark 2 *You can choose to answer or not the optional question 4, which is given in the next page.*

[9] Precise the corresponding unit (years, bps or %) for each metric.

[10] You can use any numerical nonlinear solvers in Questions 3.(c), 3.(d) and 3.(e), not necessarily a QP solver.

Optional Question

We note $x = (x_1, \ldots, x_n)$ the portfolio and $b = (b_1, \ldots, b_n)$ the benchmark. Let $m = (m_1, \ldots, m_n)$ be the vector of metrics. We remind the following properties:

$$\begin{aligned} \sum_{i=1}^{n} (x_i - b_i)^2 \cdot m_i &= (x - b)^\top M_1 (x - b) \\ \left(\sum_{i=1}^{n} (x_i - b_i) \cdot m_i\right)^2 &= (x - b)^\top M_2 (x - b) \end{aligned}$$

where $M_1 = \operatorname{diag}(m_1, \ldots, m_n)$ and $M_2 = mm^\top$. We also notice that we can always write a partial sum as a total sum:

$$\sum_{i \in \Omega} y_i = \sum_{i=1}^{n} \mathbf{1}\{i \in \Omega\} \cdot y_i = \mathbf{e}_\Omega^\top y = y^\top \mathbf{e}_\Omega$$

where $\mathbf{e}_\Omega$ is a $n \times 1$ vector such that:

$$\mathbf{e}_{\Omega,i} = \begin{cases} 1 \text{ if } i \in \Omega \\ 0 \text{ if } i \notin \Omega \end{cases}$$

4. Write each function $\mathcal{R}_{\text{AS}}(x \mid b)$, $\mathcal{R}_{\text{MD}}(x \mid b)$, $\mathcal{R}_{\text{DTS}}(x \mid b)$ in a quadratic form:

$$\mathcal{R}_{\text{Metric}}(x \mid b) = \frac{1}{2} x^\top Q_{\text{Metric}} x - x^\top R_{\text{Metric}} + c_{\text{Metric}}$$

   where c_{Metric} is a constant that does not depend on x. We note $(Q_{\text{AS}}, R_{\text{AS}}, c_{\text{AS}})$, $(Q_{\text{MD}}, R_{\text{MD}}, c_{\text{MD}})$, and $(Q_{\text{DTS}}, R_{\text{DTS}}, c_{\text{DTS}})$ the corresponding solutions. Give then the QP form:

$$\begin{aligned} x^\star &= \frac{1}{2} x^\top Q x - x^\top R \\ &\text{s.t.} \begin{cases} Ax = B \\ Cx \leq D \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{cases} \end{aligned}$$

   of the optimization problem:

$$\begin{aligned} x^\star &= \arg\min \varphi_{\text{AS}} \mathcal{R}_{\text{AS}}(x \mid b) + \varphi_{\text{MD}} \mathcal{R}_{\text{MD}}(x \mid b) + \varphi_{\text{DTS}} \mathcal{R}_{\text{DTS}}(x \mid b) \\ &\text{s.t.} \begin{cases} \sum_{i=1}^{n} x_i = 1 \\ \boldsymbol{\mathcal{CI}}(x) \leq (1 - \boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{CI}}(b) \\ 0 \leq x_i \leq 1 \end{cases} \end{aligned}$$