A Primer on Alternative Risk Premia*

Rayann Hamdan
Quantitative Research
Lyxor Asset Management, Paris
rayann.hamdan@lyxor.com

Fabien Pavlowsky
Hedge Fund Research
Lyxor Asset Management, New York
fabien.pavlowsky@lyxor.com

Thierry Roncalli
Quantitative Research
Lyxor Asset Management, Paris
thierry.roncalli@lyxor.com

Ban Zheng
Quantitative Research
Lyxor Asset Management, Paris
ban.zheng@lyxor.com

April 2016

Abstract

The concept of alternative risk premia can be viewed as an extension of the factor investing approach. Factor investing is a term that is generally dedicated to long-only equity risk factors. A typical example is the value equity strategy. Alternative risk premia designate non-traditional risk premia other than long exposure to equities and bonds. They may concern equities, rates, credit, currencies or commodities and correspond to long/short portfolios. For instance, the value strategy can be extended to credit, currencies and commodities. This paper provides an overview of the different alternative risk premia to be found in the academic and professional spheres. Using a database of commercial indices, we estimate the generic cumulative returns of 59 alternative risk premia in order to analyze their risk, diversification power and payoff function. From this, it is clear that the term "*alternative risk premia*" encompasses two different types of risk factor: skewness risk premia and market anomalies. We then reconsider portfolio allocation in light of this framework. Indeed, we show that skewness aggregation is considerably more complex than volatility aggregation, and we illustrate that the volatility risk measure is less appropriate and pertinent when managing a portfolio with these risk premia. The development of alternative risk premia shall also affect the risk/return analysis of non-linear strategies, e.g. hedge fund strategies. In particular, using alternative risk factors instead of traditional risk factors leads to an extension of the alternative beta framework. Therefore, we apply the previously estimated risk premia to a universe of hedge fund indices. To that end, we develop a model selection based on the lasso regression to identify the most pertinent risk premia for each hedge fund strategy. It appears that many traditional risk factors, with the exception of long equity and credit exposure on developed markets, vanish when we include alternative risk premia.

Keywords: Alternative risk premium, factor investing, skewness risk premium, market anomaly, risk factor, carry, event, growth, liquidity, low beta, low volatility, momentum, quality, reversal, value, short volatility, size, skewness, drawdown, diversification, portfolio allocation, option profile, alternative beta, hedge funds.

JEL classification: C50, C60, G11.

*The authors are profoundly grateful to Grégoire Vidal for his support with data management and Jean-Charles Richard for his help on the graph database. Correspondence should be addressed to thierry.roncalli@lyxor.com.

1 Introduction

Over the last few years, factor investing has grown in popularity and rapidly attracted asset managers and large institutional investors. Factor investing, together with alternative-weighted indexation, defines the smart beta approach and is generally relevant to equity investment portfolios. The underlying idea is to capture equity risk factors, such as carry, low beta, momentum, quality, value and size. These risk factors are inconsistent with the CAPM theory (Sharpe, 1964), which defines a single risk premium. But since CAPM was introduced, academic research has put forward convincing evidence that there are other systematic sources of return (Fama and French, 1992; Carhart, 1997). The concept of alternative risk premia extends the factor investing approach by considering asset classes other than equities. For instance, the carry risk factor[1] can be defined using rates, currencies or commodities. When applied to the volatility asset class, it corresponds to the short volatility strategy.

Defining an alternative risk premium is a complex task. In a strict sense, an alternative risk premium is a non-traditional risk premium[2]. In practice, alternative risk premia are systematic risk factors that can help to explain the past returns of diversified portfolios. They may be risk premia in a strict sense, but also market anomalies or common strategies. One example is the momentum strategy, which is a market anomaly and not a risk premium. Others risk factors are more difficult to classify. For instance, the volatility carry risk factor, which corresponds to the short volatility strategy, is considered as a risk premium or a market anomaly by different investors. In this article, we adopt accepted market practice and use the term "*alternative risk premia*" to denote all systematic risk factors that have resulted in positive performance in the past.

Unlike traditional risk premia, it is extremely difficult to isolate the performance of alternative risk premia. The reason is that this performance is dependent on several parameters, such as the asset universe, scoring method, weighting approach and trading implementation. This is why most academics and professional researchers devise their own backtests. In this article, we take another path: we compile a comprehensive database of about 2 000 commercial indices and investment products, to replicate alternative risk premia. By applying discretionary due diligence and quantitative filtering, we are able to estimate the generic cumulative returns of 59 alternative risk premia. This estimation procedure shows that some alternative risk premia are difficult to harvest, because investment products do not exist. Using these generic risk premia, we can analyze them with a view to creating a more diversified portfolio than a portfolio of only traditional risk premia. In particular, we explore the nature of the relationship between managing volatility and skewness risks. The second main application of these generic risk premia is the analysis of hedge fund strategies based on the alternative beta approach.

This article is organized as follows. In section two, we review the different concepts related to alternative risk premia. Namely, these are risk factors, skewness premia, market anomalies and bad times. In section three, we present the process for identifying alternative risk premia and estimating their generic cumulative performance. Section four deals with the issue of introducing alternative risk premia into an investment universe centered around traditional assets and building a diversified portfolio. Applying alternative risk premia to the risk/return analysis of hedge fund strategies is explained in section five. Finally, section six offers our concluding remarks.

[1] In the case of equities, this factor can be captured by investing in a portfolio of high dividend yield stocks.

[2] This means that it is not a long-only exposure on equities or bonds.

2 Defining risk premia, risk factors and market anomalies

In this section, we review the different concepts related to alternative risk premia. More precisely, we clarify the definition of "*risk premium*" and explore the relationship between risk premia and risk factors. We also illustrate the blurred boundary between risk premia and market anomalies. We adopt a very simple approach by focusing on the most important results, and refer to Cochrane (2001) and Martellini and Milhau (2015) for a comprehensive treatment of these topics.

2.1 Defining the risk premium in the CAPM

The capital asset pricing model (CAPM) was introduced by Sharpe in 1964, and may be viewed as an equilibrium model based on the framework defined by Markowitz (1952). In his paper, Markowitz developed the efficient frontier concept, i.e. the set of optimal mean-variance portfolios. Later, Tobin (1958) showed that the efficient frontier becomes a straight line in the presence of a risk-free asset. Contrary to Markowitz' results, it can then be shown that there is only one optimal portfolio of risky assets, which is called the tangency portfolio. One of the issues is that the tangency portfolio depends on the parameter values[3] of each investor. This was partially solved by Sharpe (1964). Using a defined set of assumptions[4], he showed that the tangency portfolio corresponds to the market-capitalization portfolio (or market portfolio). He then deduced the now familiar relationship between the expected return of asset i and the expected return of the market portfolio:

$$\mathbb{E}\left[R_i\right] - R_f = \beta_i^{mkt}\left(\mathbb{E}\left[R_{mkt}\right] - R_f\right) \tag{1}$$

where R_i and R_{mkt} are the asset and market returns, R_f is the risk-free rate and the coefficient β_i^{mkt} is the beta of asset i with respect to the market portfolio:

$$\beta_i^{mkt} = \frac{\operatorname{cov}\left(R_i, R_{mkt}\right)}{\sigma^2\left(R_{mkt}\right)}$$

Contrary to idiosyncratic risks, systematic risk cannot be diversified and investors are compensated for taking this risk. This means that the market risk premium is positive ($\pi_{mkt} = \mathbb{E}\left[R_{mkt}\right] - R_f > 0$) whereas the expected return on idiosyncratic risk is equal to zero. By definition, the idiosyncratic risk of asset i is equal to:

$$\epsilon_i = \left(R_i - R_f\right) - \beta_i^{mkt}\left(\mathbb{E}\left[R_{mkt}\right] - R_f\right)$$

where $\mathbb{E}\left[\epsilon_i\right] = 0$. This idiosyncratic risk is not rewarded because it can be hedged.

The uniqueness of the market risk premium is difficult to verify in practice, however, because it assumes that there is a single market portfolio composed of all financial assets and there is no mention of asset classes, which are the building blocks of strategic asset allocation. Therefore, long-term investors use a modified version of the CAPM and believe that the model holds at the asset class level and not at the global level. This is why they do not compute one risk premium, but several risk premia depending on the segmentation of their asset universe[5].

[3] These are the expected returns and the covariance matrix of the assets.

[4] For instance, investors have homogeneous beliefs and the market is efficient.

[5] Typical examples of asset class segmentation are: American equities, European equities, Japanese equities, emerging markets equities, sovereign bonds, corporate bonds, high yield bonds and emerging market bonds.

Summary 1 *In the CAPM, there is a single risk premium. It is equal to the excess return of the market portfolio with respect to the risk free asset. This risk premium is a compensation for being exposed to the non-diversifiable risk. In practice, investors consider several risk premia, that is one risk premium for each asset class.*

2.2 Selecting the risk factors in the arbitrage pricing theory

Ross (1976) proposed an alternative model to the CAPM, which he called arbitrage pricing theory (APT). In this model, the return on asset i is driven by a standard linear factor model:

$$R_i = \alpha_i + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \mathcal{F}_j + \varepsilon_i \tag{2}$$

where α_i is the intercept, β_i^j is the sensitivity of asset i to factor j and $\mathcal{F}_j$ is the (random) value of factor j. ε_i is the idiosyncratic risk of asset i, implying that $\mathbb{E}[\varepsilon_i] = 0$, $\operatorname{cov}(\varepsilon_i, \varepsilon_k) = 0$ for $i \neq k$ and $\operatorname{cov}(\varepsilon_i, \mathcal{F}_j) = 0$. Using arbitrage theory, we can show that the expected return of asset i is a linear function of the expected returns of the factors:

$$\mathbb{E}[R_i] - R_f = \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \left(\mathbb{E}[\mathcal{F}_j] - R_f\right) \tag{3}$$

The underlying idea of APT is that systematic risks are not entirely captured by a single market risk figure. Unlike CAPM, which relies on the validity of the Markowitz model[6], APT does not assume a specific utility function. However, it assumes that it is possible to select from a large number of assets to build a portfolio that is sufficiently diversified with no specific risk in respect of individual assets.

APT and CAPM are very different. The market risk premium in CAPM is deduced from an equilibrium argument, implying that the one-factor model is a consequence of the existence of the risk premium:

$$R_i = \alpha_i + \beta_i^{mkt} R_{mkt} + \varepsilon_i \tag{4}$$

where $\alpha_i = \left(1 - \beta_i^{mkt}\right) R_f$ and $\varepsilon_i = \epsilon_i - \beta_i^{mkt}\left(R_{mkt} - \mathbb{E}[R_{mkt}]\right)$ is a white noise process[7]. In APT, the risk model is determined ex-ante meaning that Equation (3) is deduced from the model described in Equation (2). However, this model says nothing about the sign of the excess return $\pi(\mathcal{F}_j) = \mathbb{E}[\mathcal{F}_j] - R_f$. $\pi(\mathcal{F}_j)$ is generally misinterpreted as a risk premium. Indeed, the issue is that the value taken by $\pi(\mathcal{F}_j)$ is exogenous. It can be positive, but it can also be negative, zero or even undefined.

One example of the APT framework is the four-factor model of Carhart (1997):

$$R_i = \alpha_i + \beta_i^{mkt}\left(R_{mkt} - R_f\right) + \beta_i^{smb} R_{smb} + \beta_i^{hml} R_{hml} + \beta_i^{wml} R_{wml} + \varepsilon_i \tag{5}$$

where R_{smb} is the return on small stocks minus the return on large stocks, R_{hml} is the return on stocks with high book-to-market values minus the return on stocks with low book-to-market values and R_{wml} is the return difference of winner and loser stocks over the past twelve months. This model is an extension of the three-factor model of Fama and French (1993) and has become standard in the asset management industry since its publication.

[6]This implies that investors adopt a mean-variance analysis.

[7]ϵ_i is a new form of idiosyncratic risk.

The selection of risk factors is the key issue in APT models. There are several ways to proceed, but the general idea is to test the pertinence of the risk factor $\mathcal{F}_{n_\mathcal{F}+1}$ with respect to a set of pre-existing risk factors $(\mathcal{F}_1, \ldots, \mathcal{F}_{n_\mathcal{F}})$ by comparing the model with and without the additional risk factor. Let $\alpha_i(n_\mathcal{F})$ be the alpha associated to the model with $n_\mathcal{F}$ factors. $\alpha_i(n_\mathcal{F})$ is estimated using the following time-series regression[8]:

$$R_{i,t} = \alpha_i(n_\mathcal{F}) + \sum_{j=1}^{n_\mathcal{F}} \beta_i^j \mathcal{F}_{j,t} + \varepsilon_{i,t}$$

We can also compute the corresponding R-squared statistic $\mathbf{R}^2(n_\mathcal{F})$. In the case of the $n_\mathcal{F}+1$-factor model, we have:

$$R_{i,t} = \alpha_i(n_\mathcal{F}+1) + \sum_{j=1}^{n_\mathcal{F}} \beta_i^j \mathcal{F}_{j,t} + \beta_i^{n_\mathcal{F}+1} \mathcal{F}_{n_\mathcal{F}+1,t} + \varepsilon_{i,t}$$

The relevance of the factor $\mathcal{F}_{n_\mathcal{F}+1}$ can initially be measured by calculating the gap between $\mathbf{R}^2(n_\mathcal{F})$ and $\mathbf{R}^2(n_\mathcal{F}+1)$. A more popular approach is to test the assumptions $\mathcal{H}_1 : \alpha_i(n_\mathcal{F}) \neq 0$ and $\mathcal{H}_2 : \alpha_i(n_\mathcal{F}+1) = 0$. The joint hypotheses $(\mathcal{H}_1, \mathcal{H}_2)$ implies that the set of pre-existing risk factors $(\mathcal{F}_1, \ldots, \mathcal{F}_{n_\mathcal{F}})$ is not sufficient to eliminate the alpha of portfolios, which is not the case if we include the risk factor $\mathcal{F}_{n_\mathcal{F}+1}$. The F-test of Gibbons *et al.* (1989) is also frequently used to determine whether all intercepts $\alpha_i(n_\mathcal{F}+1)$ are zero for $i = 1, \ldots, n$. The goal of all these procedures is to verify that the supplementary risk factor $\mathcal{F}_{n_\mathcal{F}+1}$ can help us to better understand the dispersion of returns. In particular, if the set of risk factors $(\mathcal{F}_1, \ldots, \mathcal{F}_{n_\mathcal{F}})$ is exhaustive, this implies that other risk factors can be defined with respect to this pre-existing set and we have:

$$\mathcal{F}_{n_\mathcal{F}+1,t} \approx \sum_{j=1}^{n_\mathcal{F}} \zeta^j \mathcal{F}_{j,t}$$

Summary 2 *APT is an arbitrage model in which idiosyncratic risks are not rewarded. However, APT does not identify systematic risk factors or determine whether or not a particular risk factor entails a risk premium. In particular, a positive estimate of excess return does not necessarily imply that the risk factor entails a risk premium.*

2.3 Estimating the excess return on risk factors

Let $\mathcal{F}_{j,t}$ be the value of the risk factor j at time t. The first approach for estimating the expected return on the risk factor is to consider the empirical mean of the sample:

$$\hat{\mu}(\mathcal{F}_j) = \frac{1}{T} \sum_{t=1}^{T} \mathcal{F}_{j,t} \tag{6}$$

We can define an estimated risk premium by subtracting the risk-free rate:

$$\hat{\pi}(\mathcal{F}_j) = \hat{\mu}(\mathcal{F}_j) - R_f$$

The fact that $\hat{\pi}(\mathcal{F}_j) > 0$ does not prove that the factor $\mathcal{F}_j$ offers a risk premium. Indeed, this result may be dependent on the study period and the risk factor may exhibit a positive return for other reasons.

[8] Here, the subscript i generally corresponds to portfolios (and not assets).

The naive approach (6) is generally not used by academics because this estimator has poor properties when the sample size is small. They prefer to follow the Fama-MacBeth approach, which accounts for the covariance structure of the factor with other risk factors and assets. The Fama-MacBeth method is a two-step procedure described as follows:

1. For each asset or portfolio i, we estimate the coefficients $\left(\beta_i^1, \ldots, \beta_i^{n_{\mathcal{F}}}\right)$ by applying a linear regression model to the time-series returns of $R_{i,t}$:

$$R_{i,t} = \alpha_i + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \mathcal{F}_{j,t} + \varepsilon_{i,t}$$

   We note $\left(\hat{\alpha}_i, \hat{\beta}_i^1, \ldots, \hat{\beta}_i^{n_{\mathcal{F}}}\right)$ the corresponding OLS estimates.

2. For each date t, we estimate the coefficients $\left(\mu_t^1, \ldots, \mu_t^{n_{\mathcal{F}}}\right)$ by applying a linear regression model to the cross-sectional returns of $R_{i,t}$:

$$R_{i,t} = c_t + \sum_{j=1}^{n_{\mathcal{F}}} \hat{\beta}_i^j \mu_t^j + u_{i,t}$$

   We note $\left(\hat{c}_t, \hat{\mu}_t^1, \ldots, \hat{\mu}_t^{n_{\mathcal{F}}}\right)$ the corresponding OLS estimates. The estimate of the expected return $\mu\left(\mathcal{F}_j\right)$ is then the average of these OLS estimates:

$$\hat{\mu}\left(\mathcal{F}_j\right) = \frac{1}{T} \sum_{t=1}^{T} \hat{\mu}_t^j \tag{7}$$

Monte Carlo simulations show that the two estimators are very close (see Figure 59 on page 114), but they may differ substantially in practice. Whereas the first estimator only depends on the time period, the Fama-MacBeth estimator also depends on the set of portfolios. However, the most important parameter is the number of observations in the sample to measure the accuracy of the estimate. Indeed, the distribution of the empirical estimator (6) and the Fama-MacBeth estimator (7) is approximatively Gaussian and we have:

$$\hat{\mu}\left(\mathcal{F}_j\right) \sim \mathcal{N}\left(\mu\left(\mathcal{F}_j\right), \frac{\sigma^2\left(\mathcal{F}_j\right)}{T}\right)$$

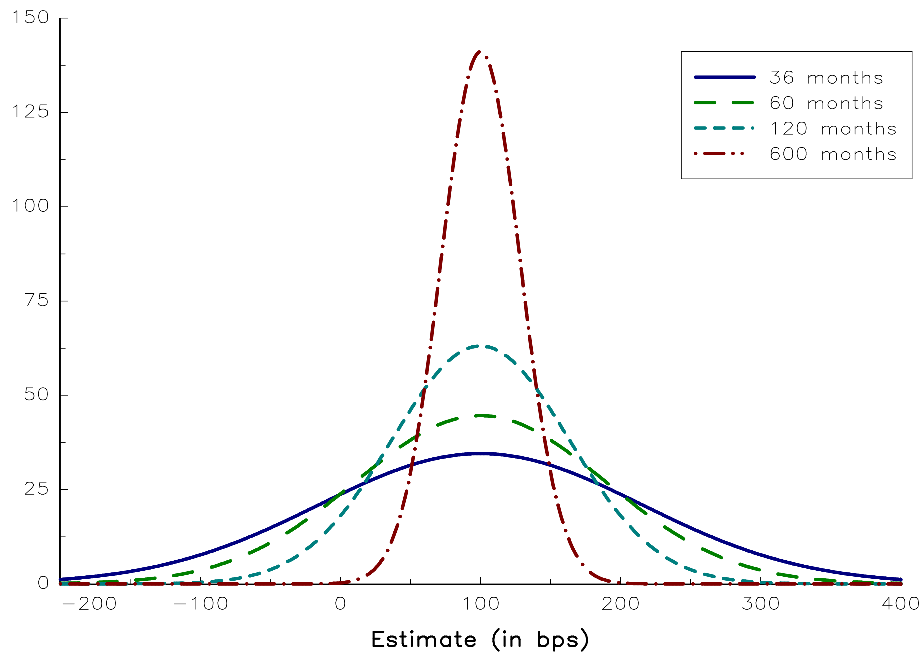
Let us consider a risk factor presenting a risk premium of 12% per year and a Sharpe ratio of 0.5. In Figure 1, we report the density of the estimator with respect to the number of observations in the sample. We notice that the estimator has a large variance even if we use 10 years of monthly data. This is an issue, because we need a long sample to precisely estimate the value of the risk premium, but we also run the risk that the risk premium level will have changed during the study period.

Summary 3 *Estimating the expected return on risk factors is critical and needs a long sample (over 10 years). It is therefore essential to determine whether the expected return level has changed during the sample period.*

2.4 Stochastic discount factor theory and bad times

In this section, we use the stochastic discount factor theory to analyze the risk premium and identify the financial assets that are positively sensitive to it.

Figure 1: Density function of the monthly risk premium



2.4.1 The source of risk premia

In the absence of arbitrage opportunities, the stochastic discount factor (or SDF) $m_{t+1} > 0$ prices all assets in the following way[9]:

$$\mathbb{E}_t \left[m_{t+1} \left(1 + R_{i,t+1} \right) \right] = 1 \tag{8}$$

where $R_{i,t+1}$ corresponds to the return on asset i between t and $t+1$. Because $R_{i,t+1} = X_{i,t+1}/P_{i,t} - 1$ where $P_{i,t}$ and $X_{i,t}$ is the price and the payoff[10] of asset i, we obtain:

$$P_{i,t} = \mathbb{E}_t \left[m_{t+1} X_{i,t+1} \right] \tag{9}$$

If we apply this equation to the risk-free asset with $P_{f,t} = 1$, we obtain $1 + R_{f,t} = 1/\mathbb{E}_t \left[m_{t+1} \right]$. Equation (9) can be rearranged so that it explicitly links the price and the expected payoff of the asset. Indeed, from the definition of the covariance operator[11], Cochrane (2001) shows that the price $P_{i,t}$ is equal to:

$$P_{i,t} = \underbrace{\frac{1}{1 + R_{f,t}} \mathbb{E}_t \left[X_{i,t+1} \right]}_{\text{Fundamental price}} + \underbrace{\operatorname{cov}_t \left(m_{t+1}, X_{i,t+1} \right)}_{\text{Risk adjustment}}$$

The first term corresponds to the "*fundamental price*" $P^{\star}_{i,t}$, which is the expected discounted payoff. As noted by Cochrane (2001), the second term is a "*risk adjustment*":

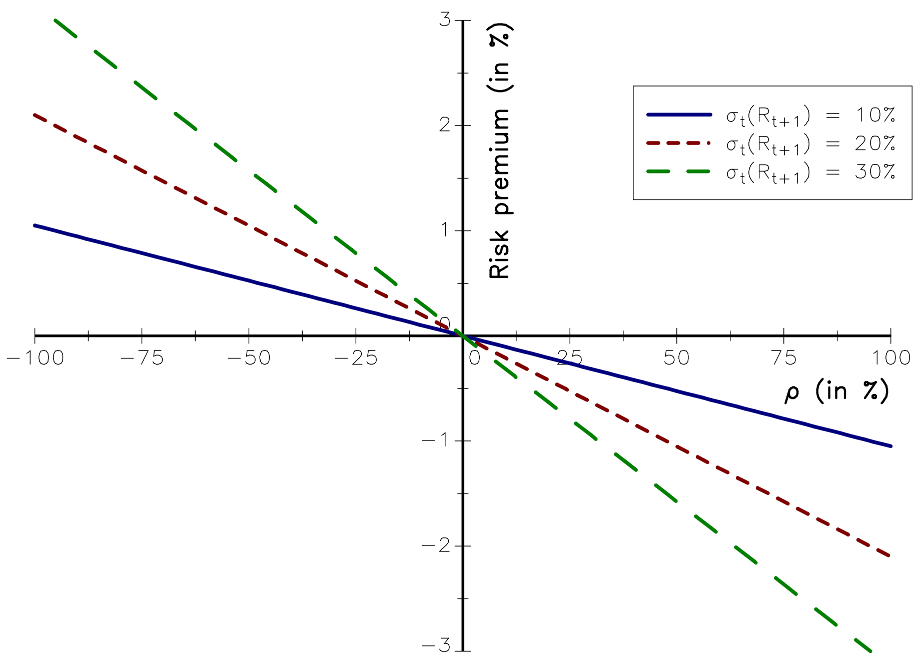
[9] $\mathbb{E}_t$ denotes the conditional expectation with respect to the information set available at time t.

[10] The payoff typically corresponds to P_{t+1} for a stock paying no dividend and $P_{t+1} + D_{t+1}$ for a stock paying a dividend.

[11] We have $\operatorname{cov}_t \left(m_{t+1}, X_{t+1} \right) = \mathbb{E}_t \left[m_{t+1} X_{i,t+1} \right] - \mathbb{E}_t \left[m_{t+1} \right] \mathbb{E}_t \left[X_{i,t+1} \right]$ meaning that $\mathbb{E}_t \left[m_{t+1} X_{i,t+1} \right] = \operatorname{cov}_t \left(m_{t+1}, X_{t+1} \right) + \mathbb{E}_t \left[m_{t+1} \right] \mathbb{E}_t \left[X_{i,t+1} \right]$.

- If m_{t+1} and $X_{i,t+1}$ are independent, the current price $P_{i,t}$ is equal to the fundamental price $P^{\star}_{i,t}$.
- A negative covariance between m_{t+1} and $X_{i,t+1}$ generates a lower price: $P_{i,t} < P^{\star}_{i,t}$.
- If the payoff $X_{i,t+1}$ has positive covariance with the stochastic discount factor, the price is higher.

Figure 2: Relationship between correlation with SDF and risk premium



If we apply Equation (9) to the excess return $R_{i,t+1} - R_{f,t}$, we get:

$$\mathbb{E}_t \left[m_{t+1} \left(R_{i,t+1} - R_{f,t} \right) \right] = 0$$

By using the same rearrangement[12] as before, it appears that the expected excess return also depends on the covariance between the stochastic discount factor and the future return:

$$\mathbb{E}_t \left[R_{i,t+1} - R_{f,t} \right] = \underbrace{-\left(1 + R_{f,t}\right) \operatorname{cov}_t \left(m_{t+1}, R_{i,t+1} \right)}_{\text{Risk adjustment}} \tag{10}$$

The risk adjustment or "*risk premium*" is positive and higher for assets that have a large negative covariance with the stochastic discount factor. Another expression of the risk

[12] We notice that $\operatorname{cov}_t \left(m_{t+1}, R_{t+1} \right) = \mathbb{E}_t \left[m_{t+1} R_{i,t+1} \right] - \mathbb{E}_t \left[m_{t+1} \right] \mathbb{E}_t \left[R_{i,t+1} \right]$. It follows that:

$$\begin{aligned} \mathbb{E}_t \left[m_{t+1} \left(R_{i,t+1} - R_{f,t} \right) \right] &= \mathbb{E}_t \left[m_{t+1} R_{i,t+1} \right] - \mathbb{E}_t \left[m_{t+1} R_{f,t} \right] \\ &= \operatorname{cov}_t \left(m_{t+1}, R_{t+1} \right) + \mathbb{E}_t \left[m_{t+1} \right] \mathbb{E}_t \left[R_{i,t+1} \right] - \mathbb{E}_t \left[m_{t+1} R_{f,t} \right] \\ &= \operatorname{cov}_t \left(m_{t+1}, R_{t+1} \right) + \mathbb{E}_t \left[m_{t+1} \right] \left(\mathbb{E}_t \left[R_{i,t+1} \right] - R_{f,t} \right) \\ &= \operatorname{cov}_t \left(m_{t+1}, R_{t+1} \right) + \frac{\mathbb{E}_t \left[R_{i,t+1} \right] - R_{f,t}}{1 + R_{f,t}} \end{aligned}$$

premium is:

$$\mathbb{E}_t\left[R_{i,t+1} - R_{f,t}\right] = -\left(1 + R_{f,t}\right)\rho_t\left(m_{t+1}, R_{i,t+1}\right)\sigma_t\left(m_{t+1}\right)\sigma_t\left(R_{i,t+1}\right) \tag{11}$$

In Figure 2, we illustrate this relationship between the correlation and the risk premium for three levels of volatility[13]. We verify that investors require a high risk premium in order to buy assets that are negatively correlated with the discount factor and are highly volatile. The key issue to understand if the asset is exposed to the risk premium is therefore its behavior with respect to the different states of the SDF. Moreover, we notice that the risk premium could vary over time as correlation and volatilities are time-varying.

Summary 4 *The stochastic discount factor theory states that the risk premium varies over time and its magnitude depends on the asset's volatility and the correlation of the asset payoff with the stochastic discount factor. In particular, the risk premium is positive if – and only if – the correlation is negative.*

2.4.2 Risk premia and bad times

Lucas (1978) suggests a general model where investors make investment choices depending on their expected utility of consumption. Investors are rational and maximize their utility function, which contains both the utility of current consumption and the expected utility of future consumption:

$$U\left(C_t, C_{t+1}\right) = u\left(C_t\right) + \delta\mathbb{E}_t\left[u\left(C_{t+1}\right)\right]$$

where $\delta \geq 0$ is the time-preference discount factor and u is the utility function. As explained by Cochrane (2001), the concavity of the utility function "*generates aversion to risk and to intertemporal substitution*", implying that investors prefer to smooth their consumption over time. The investor's objective is to maximize the utility function U under given budget constraints. In this instance, Cochrane (2001) shows that the solution satisfies this first-order condition:

$$P_{i,t} = \mathbb{E}_t\left[\delta\frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)}X_{i,t+1}\right]$$

In this model, we identify the stochastic discount factor as the marginal rate of substitution:

$$m_{t+1} = \delta\frac{u'\left(C_{t+1}\right)}{u'\left(C_t\right)}$$

Conditional to the information available at time t, we finally deduce that the risk premium is equal to:

$$\mathbb{E}_t\left[R_{i,t+1} - R_{f,t}\right] = -\kappa\rho_t\left(u'\left(C_{t+1}\right), R_{i,t+1}\right)\sigma_t\left(u'\left(C_{t+1}\right)\right)\sigma_t\left(R_{i,t+1}\right)$$

where κ is a positive scalar[14]. As explained by Cochrane (2001), the consumption model helps to understand the basic mechanisms of risk premia:

- In bad times, investors decrease their consumption and the marginal utility is high[15]. Therefore, investors agree to pay a high price for an asset that helps to smooth their consumption. To hedge bad times, investors can use assets with a low or negative risk premium. They will invest in assets that are positively correlated with these bad times only if their risk premium is high.

[13] We assume that $R_{f,t} = 5\%$ and $\sigma_t\left(m_{t+1}\right) = 10\%$.

[14] We have

$$\kappa = \frac{\delta\left(1 + R_{f,t}\right)}{u'\left(C_t\right)} \geq 0$$

[15] Because we have $\partial_c u\left(c\right) > 0$ and $\partial_c^2 u\left(c\right) < 0$.

- In good times, investors increase their consumption and we have:

$$C_{t+1} \geq C_t \geq C_{t-1} \Rightarrow m_{t+1} \geq m_t$$

In this instance, investors may select an asset for a price above its fair value if the asset is an insurance against bad times. An asset that pays off well in good times is less appealing, implying that the investor requires a higher risk premium.

This model explains why equities' risk premia must be higher than that of bonds. Indeed, during recessions, the marginal utility of consumption is high and investors require a lower price or a higher risk premium to invest in equities.

Summary 5 *In the consumption-based model, the risk premium is a compensation for accepting risk in bad times.*

2.4.3 Relationship with risk factor models

Following Cochrane (2001), Equation (10) can be rewritten as follows[16]:

$$\begin{aligned}
\mathbb{E}_t\left[R_{i,t+1}\right] &= R_{f,t} - \left(1 + R_{f,t}\right) \operatorname{cov}_t\left(m_{t+1}, R_{i,t+1}\right) \\
&= R_{f,t} - \frac{\operatorname{cov}_t\left(m_{t+1}, R_{i,t+1}\right)}{\operatorname{var}_t\left(m_{t+1}\right)} \left(\frac{\operatorname{var}_t\left(m_{t+1}\right)}{\mathbb{E}_t\left[m_{t+1}\right]}\right) \\
&= R_{f,t} - \beta\left(R_{i,t+1} \mid m_{t+1}\right) \cdot \pi\left(m_{t+1}\right)
\end{aligned}$$

where $\beta\left(R_{i,t+1} \mid m_{t+1}\right)$ is the beta of the asset return $R_{i,t+1}$ with respect to the discount factor m_{t+1}. We can interpret $\pi\left(m_{t+1}\right)$ as the risk premium. Let us decompose the return $R_{i,t+1}$ as follows:

$$R_{i,t+1} = \tilde{R}_{i,t+1} + \varepsilon_{i,t+1}$$

where $\tilde{R}_{i,t+1}$ is the projection of $R_{i,t+1}$ on m_{t+1}. By construction, the residual $\varepsilon_{i,t+1}$ satisfies $\operatorname{cov}\left(\varepsilon_{i,t+1}, m_{t+1}\right) = 0$. It follows that:

$$\mathbb{E}_t\left[R_{i,t+1}\right] = R_{f,t} - \beta\left(\tilde{R}_{i,t+1} \mid m_{t+1}\right) \cdot \pi\left(m_{t+1}\right)$$

We conclude that only the systematic risk $\tilde{R}_{i,t+1}$ is rewarded and not the specific risk $\varepsilon_{i,t+1}$.

It is no coincidence that the previous results are similar to those obtained in the CAPM. Indeed, Cochrane (2001) shows that a linear factor model can be expressed as a stochastic discount factor, implying that[17]:

$$m_{t+1} = a - b^\top \mathcal{F}_{t+1} \Leftrightarrow \mathbb{E}\left[R_i\right] = \alpha + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \pi_j$$

[16] We recall that $\mathbb{E}_t\left[m_{t+1}\right] = \left(1 + R_{f,t}\right)^{-1}$.

[17] We have:

$$\begin{aligned}
\mathbb{E}_t\left[R_{i,t+1}\right] &= R_{f,t} - \frac{\operatorname{cov}_t\left(a - b^\top \mathcal{F}_{t+1}, R_{i,t+1}\right)}{\mathbb{E}_t\left[m_{t+1}\right]} \\
&= R_{f,t} + \sum_{j=1}^{n_{\mathcal{F}}} b^j \frac{\operatorname{cov}_t\left(\mathcal{F}_{j,t+1}, R_{i,t+1}\right)}{\mathbb{E}_t\left[m_{t+1}\right]} \\
&= R_{f,t} + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \left(\frac{b^j \operatorname{var}_t\left(\mathcal{F}_{j,t+1}\right)}{\mathbb{E}_t\left[m_{t+1}\right]}\right) \\
&= R_{f,t} + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \pi_j
\end{aligned}$$

For instance, if we consider the CAPM, we have $m_{t+1} = a - b\left(R_{t+1}^{mkt} - R_{f,t}\right)$ and $\pi\left(m_{t+1}\right) = \mathbb{E}_t\left[R_{t+1}^{mkt} - R_{f,t}\right]$.

Summary 6 *Risk factors are related to the stochastic discount factor approach. More specifically, it is equivalent to consider a beta representation of expected returns and to postulate that the stochastic discount factor is a linear function of risk factors. In the consumption-based model, this implies that risk factors are compensated because of their risks in bad times.*

2.5 Why alternative risk premia?

The theory outlined above helps us to understand why equities as an asset class exhibit a risk premium. During bad times, the risk of holding equities generally increases and investors require a higher expected return to hold such financial assets. This argument can be extended to other asset classes, for example high yield bonds, private equity or cat bonds, because these financial assets do not help investors to smooth their consumption and wealth.

The definition of the risk premium at the asset class level is largely dependent on the capital asset pricing model of Sharpe (1964). In this approach, asset returns are explained by a single common risk factor and idiosyncratic factors. Since Fama and French published their seminal work (1992, 1993), it is widely accepted that several common risk factors may affect asset returns, especially in the universe of listed equities. For instance, we found the size effect (Banz, 1981), the value factor (Basu, 1977) and the momentum strategy (Jegadeesh and Titman, 1993). When these risk factors are coupled with the market risk factor, they form the four-factor model of Carhart (1997), which is a direct extension of the Fama-French model. These three risk factors are potential candidates for alternative risk premia. This concept refers to risk premia other than the market risk premium.

2.5.1 Are size, value and momentum factors alternative risk premia?

While there is evidence to suggest that size, value and momentum are risk factors (meaning that they help to explain the cross-section of stock returns), this does not mean that they necessarily exhibit a risk premium.

Mean-reverting versus trend-following strategies The study of these canonical strategies is of particular interest for understanding whether they exhibit a risk premium. In Appendix C.2 and C.3, we present two simplified models of reversal and trend-following strategies. In Figure 3, we show the return profile of these two strategies with respect to some parameters[18]. It is convex for the reversal strategy and concave for the trend-following strategy. This result, presented by Fung and Hsieh (2001), has been confirmed by many research articles (Potters and Bouchaud, 2006; Martin and Zou, 2012; Grebenkov and Serror, 2014). This is an example of two opposite behaviors occurring when the asset is performing poorly or when the trend is highly negative. The mean-reverting strategy suffers more because the current price moves away from the target price, while the trend-following strategy may perform due to its short exposure. These two strategies are also polar opposites with regards to other properties. For instance, the mean-reverting strategy presents a positive vega, while the performance of the trend-following strategy is negatively correlated to volatility risk. In the case of the mean-reverting strategy, the frequency of gain exceeds 50%, but the magnitude of the gain is limited. For the trend-following strategy, it is the loss that is

[18] For the reversal strategy, the default parameters are $S_0 = 80$, $\sigma_t = 20\%$, $T = 1$, $m = 1$ and $\bar{S} = 80$. For the trend-following strategies, we use the default values: $\hat{\mu}_0 = 30\%$, $\sigma = 20\%$, $T = 1$, $m = 1$ and $\lambda = 1$

Figure 3: Comparison of reversal and trend-following strategies

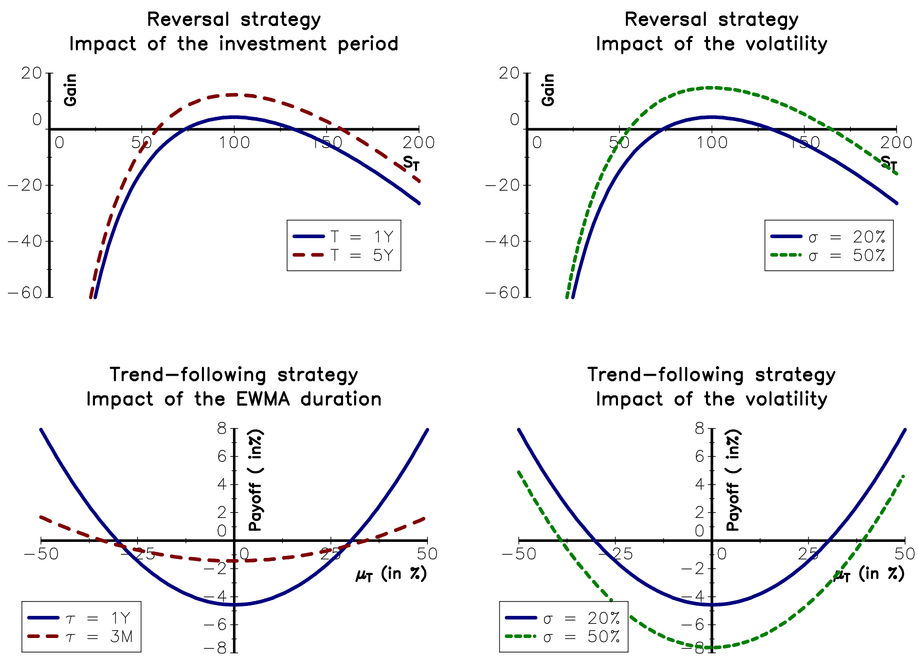
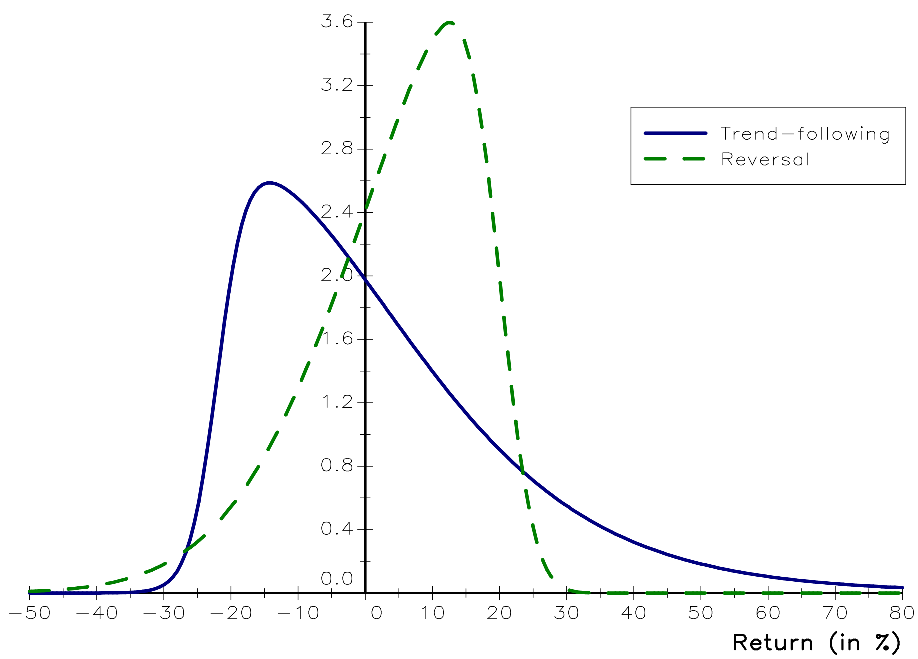


Figure 4: Probability density function of reversal and trend-following strategies



limited. In Figure 4, we summarize these different properties by referring to the probability distribution of the P&L. We observe that the fundamental difference between these two types of strategy is the skewness pattern. The skewness is positive for trend-following strategies, whereas it is negative for mean-reverting strategies[19], implying that their drawdowns must be higher. This result may be related to the research of Lempérière *et al.* (2014), which concludes that the "*risk premium is strongly correlated with tail-risk skewness*".

Summary 7 *The performance of a mean-reverting strategy may be explained by a risk premium. Indeed, it is positively correlated to bad times and losses during such periods may be substantial. Conversely, the average performance of a trend-following strategy is clearly not driven by a risk premium because it does not present a specific risk during bad times.*

SMB, HML and WML risk factors We can use the study discussed above to characterize well-known equity risk factors, namely the factors of size (SMB), value (HML) and momentum (WML). We consider a universe of n assets. We note $S_{i,t}$ the price of asset i and X_t the value of the portfolio. We assume that the portfolio can be described at time t by the vector of weights $(w_{1,t}, \ldots, w_{n,t})$. In this case, the value of the portfolio satisfies the following relationship:

$$\mathrm{d}X_t = \sum_{i=1}^{n} w_{i,t}\,\mathrm{d}S_{i,t}$$

Let $M_{i,t}$ be the market capitalization of stock i. We have $M_{i,t} = N_{i,t}S_{i,t}$ where $N_{i,t}$ is the number of shares outstanding for stock i. Without a loss of generality, we assume that $N_{i,t}$ remains constant over time. A size strategy consists of overweighting small cap stocks and underweighting large cap stocks. If we consider the following scheme:

$$w_{i,t} = \frac{1}{M_{i,t}}$$

we obtain:

$$\begin{aligned}\mathrm{d}X_t &= \sum_{i=1}^{n} \frac{1}{N_i S_{i,t}}\,\mathrm{d}S_{i,t} \\ &= \sum_{i=1}^{n} \frac{c_i}{S_{i,t}}\,\mathrm{d}S_{i,t}\end{aligned}$$

where $c_i = 1/N_i$ is a constant. We note that the size strategy may be viewed as a reversal strategy.

Let $B_{i,t}$ be the book value of stock i. The value strategy compares the current book-to-market ratio $b_{i,t} = B_{i,t}/M_{i,t}$ with the long-term value $\bar{b}$. Thus, Fama and French (1993) define the value strategy as a long position of stocks with high book-to-market values and a short position of stocks with low book-to-market values. Therefore, we have:

$$\begin{aligned}w_{i,t} &= b_{i,t} - \bar{b} \\ &= \frac{B_{i,t}}{N_i S_{i,t}} - \bar{b}\end{aligned}$$

[19]See Appendix C.2 and C.3 for their mathematical expressions.

Generally, book values change at a low frequency and remain constant during a fiscal quarter or fiscal year. We then obtain:

$$w_{i,t} = \frac{c_i}{S_{i,t}} - \bar{b}$$

where $c_i = B_i/N_i$ is a constant. Again, the value strategy is a reversal strategy.

Jegadeesh and Titman (1993) define the WML risk factor as the performance of a portfolio that is long in winner stocks and short in loser stocks. The trend is simply measured by the performance over the past twelve months and can be approximated by an exponential weighted moving average with a low decay factor τ. From a theoretical point of view, the WML risk factor is close to the trend-following strategy presented in Appendix C.3. Let us consider a strategy, where wealth is defined as follows:

$$\frac{\mathrm{d}X_t}{X_t} = \sum_{i=1}^{n} w_{i,t} \frac{\mathrm{d}S_{i,t}}{S_{i,t}}$$

The momentum factor may be defined by the following allocation process:

$$w_{i,t} = \hat{\mu}_{i,t}$$

Using the results in Appendix C.3, we conclude that the momentum strategy is a concave payoff and exhibits a positive skewness.

If the trend of the stock is negative, the momentum strategy will be short in the stock. In a sense, the momentum strategy follows the market. Conversely, if the price of the stock decreases, exposure is increased within the size and value strategies. In this case, they go against the market. They then face a distressed risk, typically liquidity risk for the size strategy and default risk for the value strategy[20]. We notice that these risks are positively correlated with bad times. On the other hand, the momentum strategy does not entail particularly high exposure to a specific risk occurring systematically when the investor falls upon hard times. Therefore, the performance of such strategies is explained by factors other than a risk premium.

Summary 8 *The performance of size and value risk factors may be explained by a risk premium, but this is not the case for the momentum risk factor.*

2.5.2 Market anomalies and behavior theory

The positive performance of the momentum risk factor is generally explained by the theory of behavioral finance (Barberis *et al.*, 1998; Hirshleifer, 2001). In this case, the momentum pattern may be explained by either an under-reaction to earnings announcements and news, a delayed reaction, excessive optimism or pessimism, etc. (Barberis and Thaler, 2003). The behavioral theory posits that there are many investment strategies that can exhibit good performance and do not have a risk premium. The popularity of these strategies and the long track record of their performance can encourage us to classify them as risk

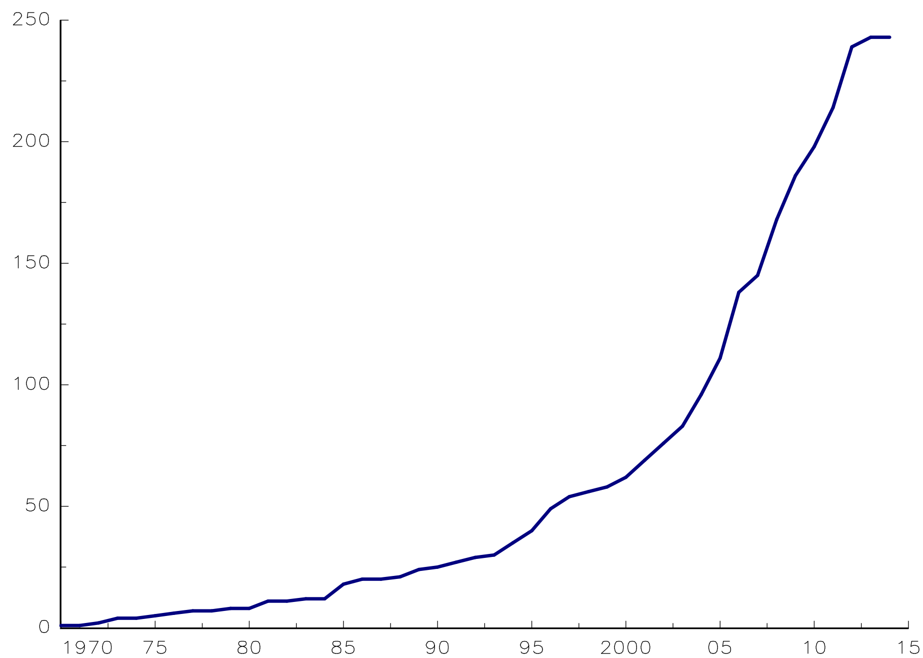
[20]The value strategy is exposed to the default risk because we have:

$$\lim_{S_{i,t}\to 0} w_{i,t} = \lim_{S_{i,t}\to 0} \frac{B_{i,t}}{M_{i,t}} = \infty$$

This means that a stock whose price goes to zero has an attractive book-to-market ratio. The strategy does not therefore distinguish between stocks that will revert to their fundamental price and stocks that will default.

premia. For instance, this applies to momentum strategies, but also to low beta or quality portfolios. As an illustration, Grinblatt *et al.* (1995) found that "77% *of the mutual funds were momentum investors, buying stocks that were past winners*". Baker *et al.* (2011) document that low beta portfolios outperform high beta portfolios over a long period of time. It is now accepted that the strong performance of low beta or low volatility assets is explained by investors' leverage aversion (Frazzini and Pedersen, 2014). The quality strategy is another good example of market anomalies, which have delivered positive performance in recent years (Asness *at al.*, 2013; Novy-Marx, 2013). Once again, behavior theory can help to understand the profitability of such portfolios, which are clearly risk factors, but not risk premia (Hou *et al.*, 2015).

Figure 5: Cumulative number of risk factors and market anomalies published in top academic journals



This is a significant issue, because the number of market anomalies is large. As an illustration, Harvey *et al.* (2016) list more than 200 risk factors and market anomalies that explain the cross-section of equity returns and were published in top academic journals (Figure 5). By interpolating the cumulative number of factor discoveries, we find that there will be in excess of 1 000 common equity risk factors by the end of the century. This number could be even larger with the development of data mining and machine learning techniques. In this context, it is extremely difficult to make the distinction between risk factors, market anomalies and backtests. An illustration is provided by Novy-Marx (2014), who found that expected returns of stocks may be empirically explained by exotic patterns, such as the weather in Manhattan or the alignment of Mars and Saturn. In this specific case, investors will certainly agree that these results are just the product of data mining. However, with more traditional patterns, it is difficult to decide whether a backtest is a market anomaly or just a stylized fact. For instance, Rozeff and Kinney (1976) described some seasonal effects, and in particular the January effect. Since its publication, this effect has disappeared.

With behavior theory, we can always find an explanation for the strong performance of market anomalies. This is why it does not help to distinguish between true and false market anomalies. Moreover, a market anomaly may disappear or decrease in strength as it becomes known and traded. This is the main conclusion of McLean and Pontiff (2016):

> "*We study the out-of-sample and post-publication return predictability of* 97 *variables shown to predict cross-sectional stock returns. Portfolio returns are* 26% *lower out-of-sample and* 58% *lower post-publication. The out-of-sample decline is an upper bound estimate of data mining effects. [...] Post-publication declines are greater for predictors with higher in-sample returns [...] Predictor portfolios exhibit post-publication increases in correlations with other published-predictor portfolios.*"

Fortunately, some market anomalies survive, such as the low volatility anomaly. Nevertheless, their selection raises the same issue as the identification of risk premia.

Summary 9 *The momentum strategy is a market anomaly. Such anomalies are frequent in asset pricing, such as low beta and quality anomalies, and generally present good profitability*[a]*. Some of them are risk factors, but they are not risk premia. However, in accordance with market practice, we continue to use the term "alternative risk premia" to denote the set of market anomalies and risk premia.*

[a] If this were not the case, they would not be called anomalies

2.5.3 Statistical properties of alternative risk premia

The skewness premium assumption Lempérière *et al.* (2014a) propose an elegant approach to characterize the risk premium property of a strategy. Indeed, they postulate that there is a linear relationship between the Sharpe ratio of a risk premium strategy and its skewness:

$$\frac{\mu(R_t) - R_f}{\sigma(R_t)} = a - b\gamma_1(R_t) \tag{12}$$

where $\mu(R_t)$, $\sigma(R_t)$ and $\gamma_1(R_t)$ are the expected return, the volatility and the skewness of the strategy, respectively. Using a set of risk premia strategies, they estimate that $\hat{a} \approx 1/3$ and $\hat{b} \approx 1/4$. In Figure 6, we report their linear regression and the 95% confidence level. The authors assume that a risk premium strategy must have a negative skewness and belong to the region corresponding to the 95% confidence level. Based on their backtests, they exclude HML and WML risk factors as alternative risk premia.

The issue with the aforementioned statistical procedure is the fact that the results depend upon the construction of the strategy and the study period. As an illustration, we consider the US Fama-French risk factors, which can be found on the website[21] of Kenneth French. In Table 1, we report the empirical skewness for different lag windows and periods. We consider MKT, SMB, HML and WML risk factors. For the SMB, we use both the long/short (LS) and long-only (LO) portfolios. For HML and WML risk factors, we only consider long-only portfolios, but we make the distinction between big cap (BC) and small cap (SC) stocks. We observe that the skewness of the MKT risk factor is negative when we calculate it over the last 10 years. If we consider the last 20 years, only the monthly and quarterly returns exhibit a negative skewness and the probability distribution is right-skewed over the last 30 years. The sensitivity to the study period is therefore important. We also notice that the construction of the risk factor has an impact on the results. For instance, the long/short SMB risk factor has a negative skewness, whereas the long-only SMB portfolio has a positive skewness over the last 10 years.

Figure 6: Relationship between skewness and Sharpe ratio (Lempérière *et al.*, 2014a)

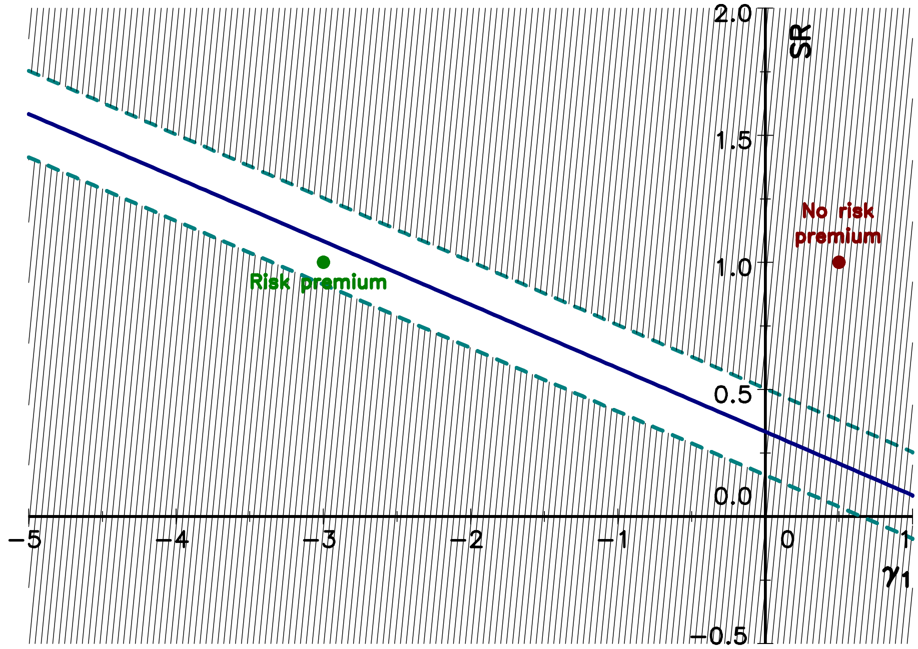
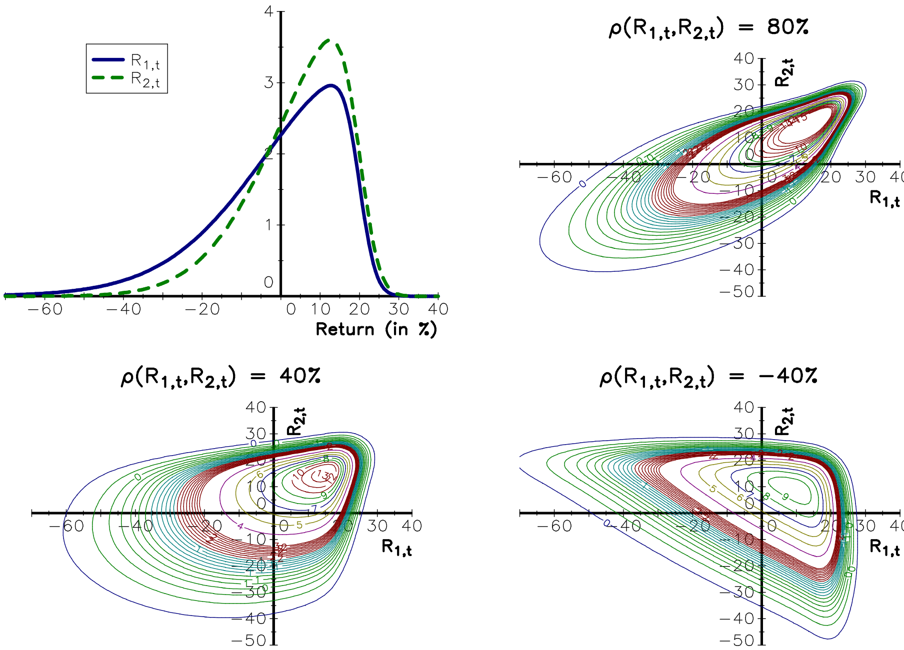


Table 1: Skewness coefficient γ_1 for different portfolios

Lag	MKT	SMB		HML		WML	
		LS	LO	LO-BC	LO-SC	LO-BC	LO-SC
				2005 − 2014			
1M	−0.74	−1.43	−0.95	−0.68	−1.36	−0.55	−0.69
3M	−0.47	−1.41	−0.39	−0.83	−0.57	−0.28	−0.03
6M	−0.16	−1.45	0.20	−0.51	−0.01	0.03	0.58
1Y	−0.22	−1.01	0.73	−0.48	0.52	−0.15	0.85
				1995 − 2014			
1M	−0.73	−0.06	−1.37	−0.80	−1.70	−0.53	−1.42
3M	−0.57	0.28	−0.38	−0.83	−0.53	−0.33	−0.29
6M	0.12	0.37	0.34	−0.18	0.04	0.28	0.49
1Y	0.15	0.66	0.97	−0.18	0.58	0.22	0.97
				1985 − 2014			
1M	0.20	−0.25	−0.46	0.01	−0.94	0.06	−0.59
3M	0.48	0.41	0.40	0.50	0.39	0.57	0.47
6M	0.66	0.63	0.78	1.00	0.87	0.74	0.85
1Y	0.73	0.40	0.84	0.95	0.87	0.71	0.83

Figure 7: The correlation problem



In Formula (12), the correlation dimension with bad times is missing. Let us consider two left-skewed strategies as shown in Figure 7. We note $R_{1,t}$ and $R_{2,t}$ their returns. We have reported the density plot of the bivariate random vector $(R_{1,t}, R_{2,t})$ by considering different hypotheses on the correlation $\rho(R_{1,t}, R_{2,t})$. When this is negative, we can ask whether the two strategies may both be risk premia, because one strategy may serve to partially hedge the other strategy. Previously, we have seen that a risk premium is negatively correlated to the stochastic discount factor or positively correlated to bad times. This gives the following situation:

	$\gamma_1(\mathcal{F}_j) < 0$	$\gamma_1(\mathcal{F}_j) > 0$
$\rho(\mathcal{F}_j, m) < 0$	✔	✘
$\rho(\mathcal{F}_j, m) > 0$	?	✘

If the risk factor has a negative skewness and negative correlation with the SDF, it may be considered as a risk premium (✔). If the skewness is positive, the risk factor is not a risk premium (✘). The case $\gamma_1(\mathcal{F}_j) < 0$ and $\rho(\mathcal{F}_j, m) > 0$ is more ambiguous (?). In this case, the risk factor may not be a risk premium, because it is not a risky strategy in bad times. Let $\mathcal{F}^\star$ be a typical risk premium. The previous correlation analysis is equivalent to the following condition: $\mathcal{F}_j$ is a risk premium if it is positively correlated to the typical risk premium $\mathcal{F}^\star$ in bad times. For instance, the typical risk premium in equities is the market risk premium. This implies that equity alternative risk premia must be correlated to the market risk premium, for example during the 2008 financial crisis.

Summary 10 *The risk premium of a strategy is an increasing function of the volatility and a decreasing function of the skewness. However, the statistical identification of alternative risk premia is an issue because these relationships depend on the sample period.*

[21] `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`

The drawdown premium assumption In the previous paragraph, we explored the relationship between a risk premium and skewness risk. However, this measure of risk is not easy for investors to understand. Moreover, significant negative skewness is generally associated with a high probability of a large drawdown. This is why investors prefer to consider the maximum drawdown, which represents the maximum loss (or the peak-to-valley return) over a specified time period. Equation (12) may then be replaced by a positive relationship between the excess return $\pi(R_t) = \mu(R_t) - R_f$ and the (future) drawdown DD_t:

$$\pi(R_t) = f(\mathrm{DD}_t) \tag{13}$$

In this case, an investor requires a high risk premium to invest in a portfolio if its drawdown is high (and the opposite is also true):

$$\mathrm{DD}_t > \mathrm{DD}'_t \Longrightarrow \pi(R_t) > \pi(R'_t)$$

The drawdown premium assumption is therefore a new formulation of the consumption-based model. However, Formula (13) is not sufficient to identify a risk premium, because it must also be correlated to bad times. This means that drawdowns of risk premia tend to occur at the same time[22]. In order to understand this correlation pattern, we consider the option profile associated to a specific risk premium.

In Figure 8, we report the payoff of alternative risk premia with respect to a traditional risk premium (TRP), typically a long-only equity portfolio. In this case, bad times correspond to the drawdown of the traditional risk premium. If the ARP exhibits a long call profile, this means that the loss is limited and small. This implies that the performance of this ARP can not be explained by a risk premium argument, even if it exhibits a positive excess return. The case of the short call profile is interesting, because it exhibits a drawdown when the market is up. This means that the drawdown occurs in good times and not in bad times. Again, the positive excess performance of this ARP can be explained by a market anomaly, but not by a risk premium argument. By definition, a long put profile is the opposite of a risk premium, because it is a strategy that hedges bad times. Finally, only the short put profile can be considered as a risk premium, because its drawdown occurs in bad times. It is interesting to relate this analysis to the trend-following strategy on multi-asset classes. Fung and Hsieh (2001) showed that this strategy has a long straddle option profile (Figure 9). Based on our analysis, it is obvious that this strategy is a market anomaly, because its drawdown is not correlated to bad times.

Summary 11 *Among typical option profiles, only the short put payoff may be considered as a risk premium. If a strategy exhibits a long call, short call or long put profile and has a positive excess return, it is presumably a market anomaly and not a skewness risk premium.*

3 Identification of alternative risk premia

3.1 Universe of potential candidates

In his famous book *Expected Returns*, Ilmanen (2011) presents a three-dimensional analysis of investments based on asset classes, strategy styles and risk factors[23]:

[22] This conclusion of the consumption-based model does not always hold in practice. Indeed, bad times can differ for some specific risk premia, such as cat bonds or weather derivatives. The existence of risk premia with different drivers of bad times may therefore pose as potential exception to the stochastic discount factor model.

[23] Antti Ilmanen uses the term "*risk factor*" in a slightly different way to how it is used in this paper. It denotes the economic drivers of asset class or strategy style returns.

Figure 8: Option profiles and risk premia

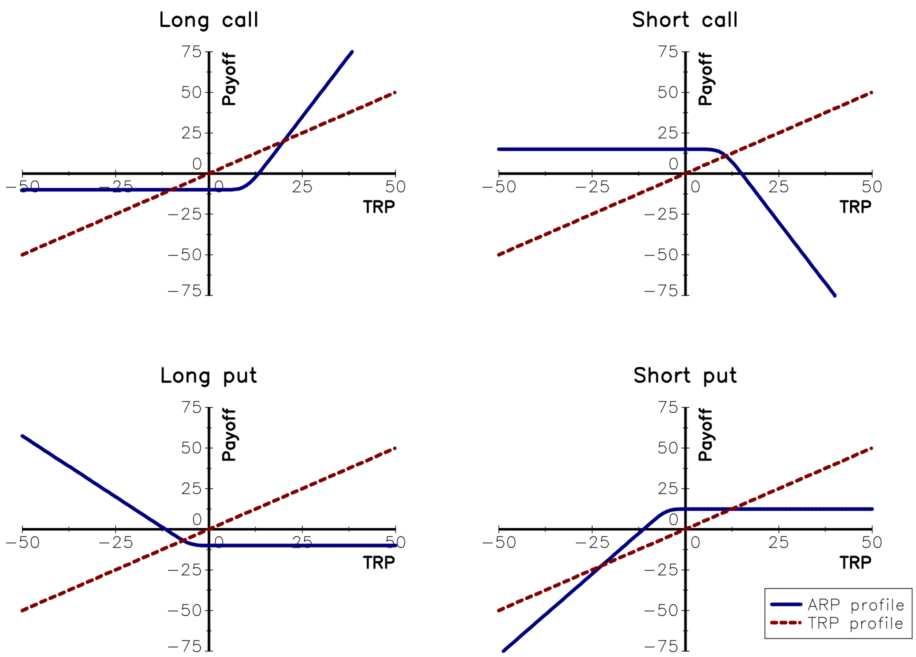
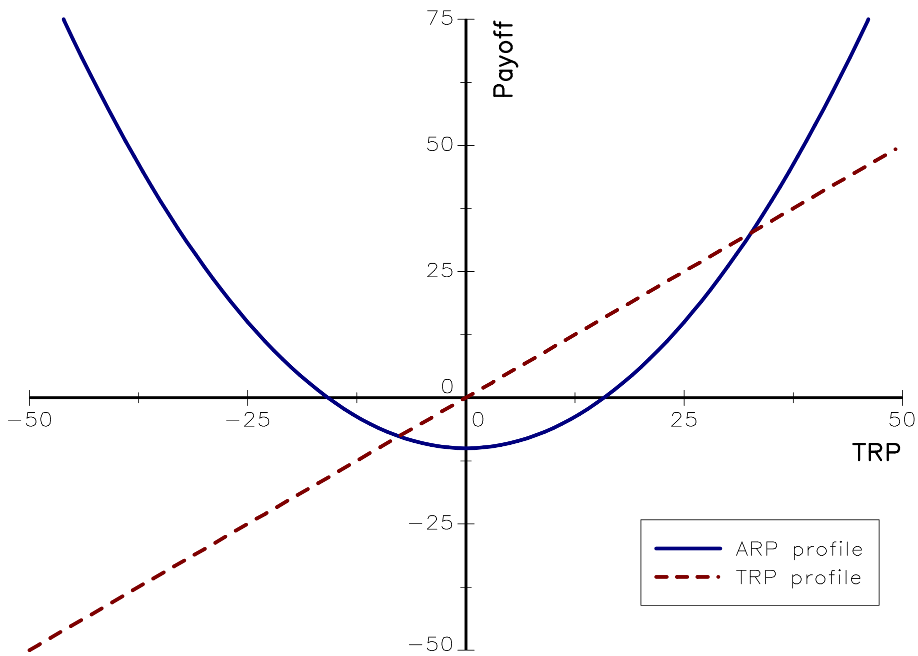


Figure 9: The case of a long straddle profile



- The asset class dimension concerns equities, rates (or government bonds), credit and many alternative ones (commodities, hedge funds, real estate and private equity).
- The strategy style dimension includes buy-and-hold, value, carry, momentum and volatility.
- The risk factor dimension focuses on economic growth, inflation, liquidity and tail risk.

This classification has been adopted by many asset owners. It is sometimes simplified to a two-dimensional analysis by only considering asset classes and risk factor strategies. The term "*risk factor strategy*" is used here to denote a potential candidate (a trading strategy or a risk factor) to be an alternative risk premium. Table 2 shows an example of such classification. We separate risk factors that can be found in several asset classes, and risk factors that are specific to equities.

Table 2: Academic mapping of ARP candidates

Strategy	Equities	Rates	Credit	Currencies	Commodities
Market			✓		✓
Carry	✔	✔	✔	✔	✔
Liquidity	✔	✔	✔	✔	✔
Momentum	✔	✔	✔	✔	✔
Reversal	✔	✔		✔	✔
Value	✔	✔	✔	✔	✔
Volatility	✔	✔	✔	✔	✔
Event	✔				
Growth	✔				
Low volatility	✔				
Quality	✔				
Size	✔				

Traditional risk premia correspond to buy-and-hold strategies on equities and bonds. They are generally implemented by regional markets (US, Europe, Japan, emerging markets, etc) and ratings categories (government bonds, corporate bonds and high yield bonds). Sometimes, investors put credit in the alternative risk premia category. The case of commodities is also interesting, because commodities may be considered either as a traditional, alternative or zero risk premium. The latter is the most accepted approach, and most institutional investors use commodities as natural hedges for inflation risk (Ang, 2014) and not for capturing a risk premium. Most pension funds include other asset classes in their strategic asset allocation, such as real estate, infrastructure and private equity.

Following the classification adopted in Table 2, the major potential alternative risk premia are listed below. However, this list is not exhaustive and does not include, for instance, insurance-linked securities, such as cat bonds or weather products.

3.1.1 Carry

The underlying idea of a carry strategy is to capture a spread or a return by betting that the underlying risk will not occur or that market conditions will stay the same. One famous example of such a strategy is the currency carry trade. It consists in being long on currencies with high interest rates and short on currencies with low interest rates. If exchange rates do not change, this portfolio generates a positive return.

Definition Koijen *et al.* (2015) extended the concept of carry strategy to other asset classes. They defined the carry of a futures (or forward) contract as the expected return if the spot price remains the same. Let X_t be the capital allocated at time t to finance a futures position on asset S_t. At time $t+1$, the excess return of this investment is[24]:

$$R_{t+1}\left(X_t\right) - R_f = \mathcal{C}_t + \frac{\mathbb{E}_t\left[\Delta S_{t+1}\right]}{X_t} + \varepsilon_{t+1}$$

where $\varepsilon_{t+1} = \left(S_{t+1} - \mathbb{E}_t\left[S_{t+1}\right]\right)/X_t$ is the unexpected price change and $\mathcal{C}_t$ is the carry:

$$\mathcal{C}_t = \frac{S_t - F_t}{X_t}$$

It follows that the expected excess return is the sum of the carry and the expected price change:

$$\mathbb{E}_t\left[R_{t+1}\left(X_t\right)\right] - R_f = \mathcal{C}_t + \frac{\mathbb{E}_t\left[\Delta S_{t+1}\right]}{X_t}$$

The nature of these two components is different. The carry is an ex-ante observable quantity whereas the expected price change depends on the dynamic model of S_t. If we assume that the spot price does not change ($\mathcal{H}$), the expected excess return is equal to the carry. This means that the carry investor will prefer asset i to asset j if the carry of asset i is higher:

$$\mathcal{C}_{i,t} \geq \mathcal{C}_{j,t} \Longrightarrow i \succ j$$

The carry strategy would then be long on high carry assets and short on low carry assets.

Remark 1 *In the case of a fully-collateralized position $X_t = F_t$, the value of the carry becomes:*

$$\mathcal{C}_t = \frac{S_t}{F_t} - 1$$

Currency carry Let us come back to the currency carry trade. Let S_t, r_t and r_t^* be the spot exchange rate, the domestic interest rate and the foreign interest rate for the period $[t, t+1]$, respectively. The forward exchange rate F_t is then equal to:

$$F_t = \frac{1 + r_t}{1 + r_t^*} S_t$$

implying that the carry is approximately equal to the interest rate differential:

$$\mathcal{C}_t = \frac{r_t^* - r_t}{1 + r_t} \simeq r_t^* - r_t$$

We verify that the carry is equal to the interest rate differential. Therefore, the carry strategy is long on currencies with high interest rates and short on currencies with low interest rates.

[24] Based on the assumption that the futures price expires at the future spot price ($F_{t+1} = S_{t+1}$), Koijen *et al.* (2015) showed that:

$$\begin{aligned} R_{t+1}\left(X_t\right) - R_f &= \frac{F_{t+1} - F_t}{X_t} \\ &= \frac{S_{t+1} - F_t}{X_t} \\ &= \frac{S_t - F_t}{X_t} + \frac{\mathbb{E}_t\left[S_{t+1}\right] - S_t}{X_t} + \frac{S_{t+1} - \mathbb{E}_t\left[S_{t+1}\right]}{X_t} \end{aligned}$$

Equity carry For equities, Koijen *et al.* (2013) showed that:

$$\mathcal{C}_t \simeq \frac{\mathbb{E}_t\left[D_{t+1}\right]}{S_t} - r_t$$

where $\mathbb{E}_t\left[D_{t+1}\right]$ is the risk-neutral expected dividend for time $t+1$. If we assume that dividends are constant, the carry is the difference between the dividend yield $\mathcal{Y}_t$ and the risk-free rate r_t:

$$\mathcal{C}_t = \mathcal{Y}_t - r_t$$

In this case, the carry strategy is long on stocks with high dividend yields and short on stocks with low dividend yields. This strategy is generally implemented in a long-only format, implying that the carry strategy consists in selecting stocks with high dividend yields. This strategy may be improved by considering forecasts of dividends. In this case, we have:

$$\frac{\mathbb{E}_t\left[D_{t+1}\right]}{S_t} = \frac{D_t + \mathbb{E}_t\left[\Delta D_{t+1}\right]}{S_t}$$

and:

$$\mathcal{C}_t = \mathcal{Y}_t + g_t - r_t$$

where g_t is the expected dividend growth. In practice, dividends are announced in advance and earnings estimates from analysts are easily available using the I/B/E/S database. This is why many carry strategies are implemented using I/B/E/S dividend yield forecasts.

Another application of the equity carry strategy concerns dividend futures. The underlying idea is to take a long position on dividend futures where the dividend premium is the highest and a short position on dividend futures where the dividend premium is the lowest. Because dividend futures are on equity indices, the market beta exposure is generally hedged.

In order to distinguish between these two carry strategies, we call them "*Carry – High Dividend Yield (HDY)*" and "*Carry – Dividend Futures (DF)*".

Fixed-income carry In the case of bonds, we generally distinguish between several forms of carry trade. In order to understand their differences, it is helpful to recall some basics about the term structure of interest rates.

The price of a zero-coupon bond with maturity date T is equal to:

$$B_t\left(T\right) = e^{-(T-t)R_t(T)}$$

where $R_t\left(T\right)$ is the corresponding zero-coupon rate. In Figure 10, we present the effects of a parallel shift, a steepening of the slope and a curvature movement on the yield curve.

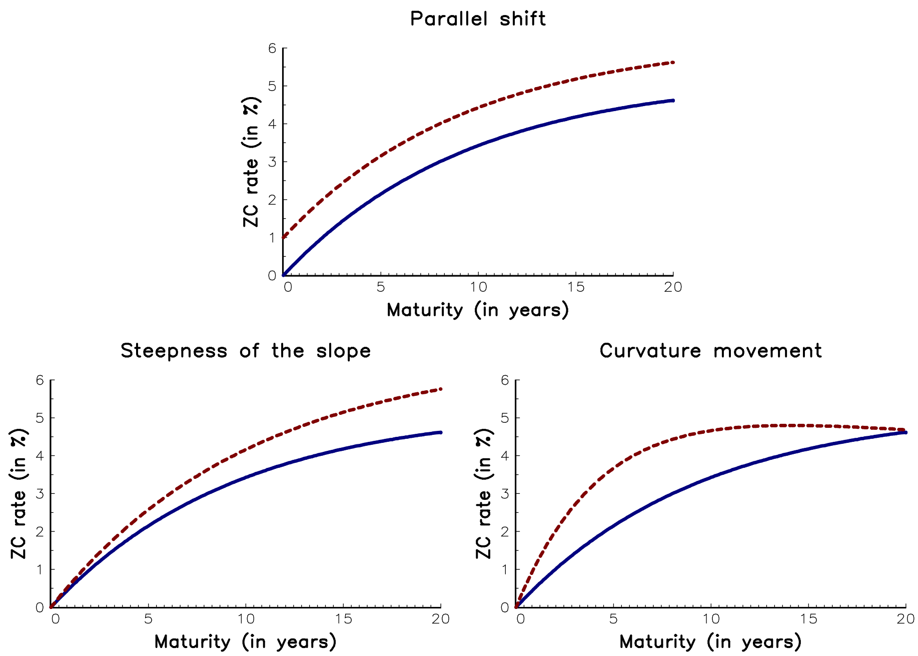
These movements impact zero-coupon rates, but also forward interest rates. Let $F_t\left(T,m\right)$ denote the forward interest rate for the period $\left[T, T+m\right]$, which is defined as follows:

$$B_t\left(T+m\right) = e^{-mF_t(T,m)}B_t\left(T\right)$$

We deduce that:

$$F_t\left(T,m\right) = -\frac{1}{m}\ln\frac{B_t\left(T+m\right)}{B_t\left(T\right)}$$

Figure 10: Movements of the yield curve



It follows that the instantaneous forward rate is given by this equation:

$$F_t(T) = F_t(T, 0) = \frac{-\partial \ln B_t(T)}{\partial T}$$

Firstly, let us consider a strategy that consists in being long the forward contract on the forward rate $F_t(T, m)$ and selling it at time $t + \mathrm{d}t$ with $t + \mathrm{d}t \leq T$. In Figure 11, we report the forward rate for different maturities and tenors. We observe that forward rates are higher than spot rates. Under the hypothesis $(\mathcal{H})$ that the yield curve does not change, rolling forward rate agreements can then capture the term premium and the roll down. We notice that the difference is higher for long maturities. However, the risk associated with such a strategy is that of a rise in interest rates. This is why this carry strategy is generally implemented by using short-term maturities (less than two years). In Appendix C.4.1, we show that the carry of such strategy is equal to:

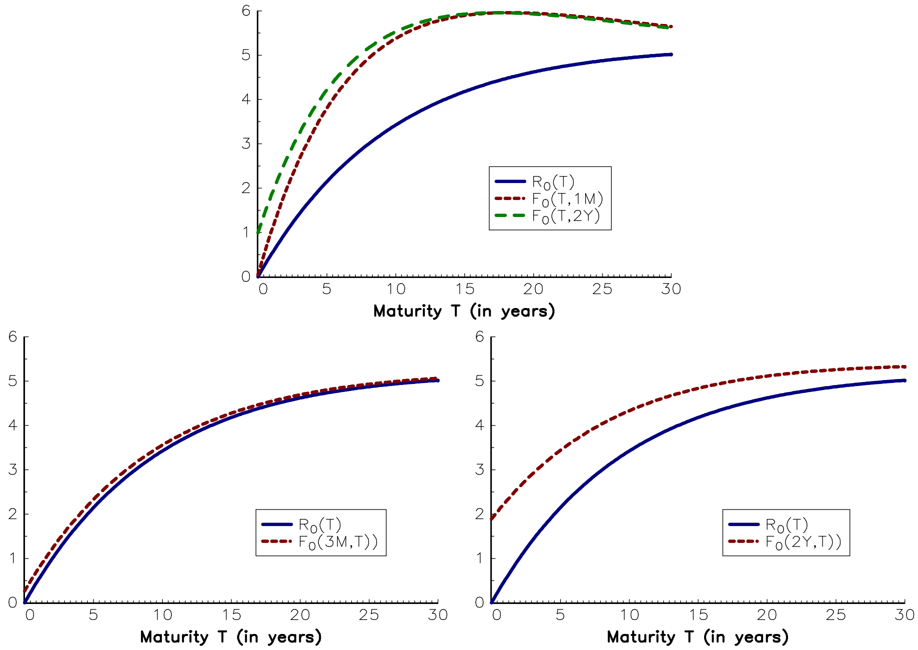
$$\mathcal{C}_t = \underbrace{R_t(T) - r_t}_{\text{term premium}} + \underbrace{\partial_{\bar{T}} \bar{R}_t(\bar{T})}_{\text{roll down}}$$

where $\bar{R}_t(\bar{T})$ is the zero-coupon rate with a constant time to maturity $\bar{T} = T - t$.

Let us now consider a second carry strategy, which corresponds to a long position in the bond with maturity T_2 and a short position in the bond with maturity T_1. The exposure of the two legs are adjusted in order to obtain a duration-neutral portfolio. This strategy is known as the slope carry trade. In Appendix C.4.2, we derive the expression of the carry and find:

$$\mathcal{C}_t = \underbrace{(R_t(T_2) - r_t) - \frac{D_2(T_1)}{D_t(T_1)}(R_t(T_1) - r_t)}_{\text{duration neutral slope}} + \underbrace{\partial_{\bar{T}} \bar{R}_t(\bar{T}_2) - \frac{D_2(T_1)}{D_t(T_1)} \partial_{\bar{T}} \bar{R}_t(\bar{T}_1)}_{\text{duration neutral roll down}}$$

Figure 11: Sport and forward interest rates



In practice, the time to maturity is set at two and 10 years. We notice, however, that the carry can take positive or negative values, implying that a systematic strategy long 10Y/short 2Y does not necessarily produce a positive return even if the term structure of interest rates is an increasing function of the maturity. This is why investors may implement this strategy by considering the yield curves of several countries. In this case, the carry strategy consists in being long on positive (or higher) slope carry and short on negative (or lower) slope carry.

In order to distinguish between these three carry strategies, we call them "*Carry – Forward Rate Bias (FRB)*", "*Carry – Term Structure Slope (TSS)*" and "*Carry – Cross Term Structures (CTS)*".

Credit carry Koijen *et al.* (2013) defined the carry of credit portfolios sorted by maturity and credit quality in the same way as for government bonds. More precisely, if we consider a long position on a credit bond and a short position on a government bond with the same duration, we obtain:

$$\mathcal{C}_t = \underbrace{\mathcal{s}_t\left(T\right)}_{\text{spread}} + \underbrace{\partial_{\bar{T}}\,\bar{R}_t^{\star}\left(\bar{T}\right) - \partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}\right)}_{\text{roll down difference}} \tag{14}$$

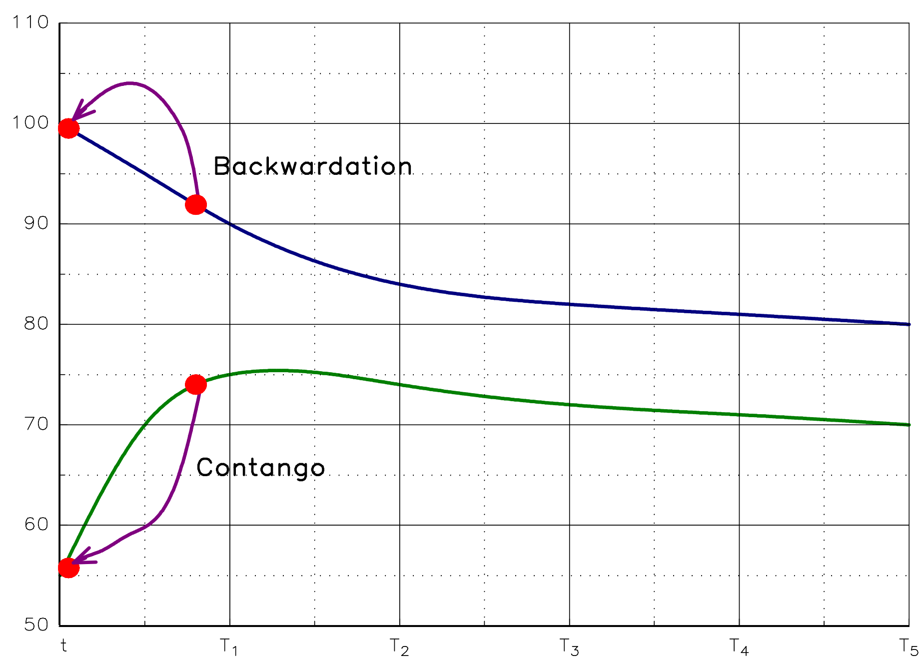
where $\mathcal{s}_t\left(T\right) = R_t^{\star}\left(T\right) - R_t\left(T\right)$ is the credit spread and $R_t^{\star}\left(T\right)$ is the yield-to-maturity of the credit bond. In fact, the carry can be approximated by the credit spread alone, because this effect is larger than the roll down effect. In practice, there are two main approaches to implementing the credit carry strategy. The first one is to build a long/short portfolio with corporate bond indices or baskets. The bond universe can be investment grade or high yield. In the case of HY bonds, the short exposure can be an IG bond index. The second approach consists in using credit default swaps (CDS). Typically, we sell credit protection on HY credit default indices (e.g. CDX.NA.HY) and buy protection on IG credit default indices (e.g. CDX.NA.IG).

Commodity carry A specific feature of the commodity asset class is the contango/backwardation aspect of the term structure of futures contracts. In Figure 12, we report these two configurations. When the term structure is in backwardation, the price of the futures contract is lower than the spot price and the carry $\mathcal{C}_t$ is positive (see Figure 12). When the term structure is in contango, we observe the opposite situation and the carry is negative. A carry strategy therefore corresponds to a long/short strategy between futures contracts with different maturities:

- In the case of backwardation, we have a long exposure on longer maturities and a short exposure on shorter maturities.
- In the case of contango, we have a long exposure on shorter maturities and a short exposure on longer maturities.

In Figure 13, we reproduce an illustration given by Roncalli (2013) concerning crude oil futures contracts. We notice that there may be a large difference between the spot price and futures contracts with long maturities. However, carry strategies reflect liquidity issues, because liquidity generally decreases with the time to maturity.

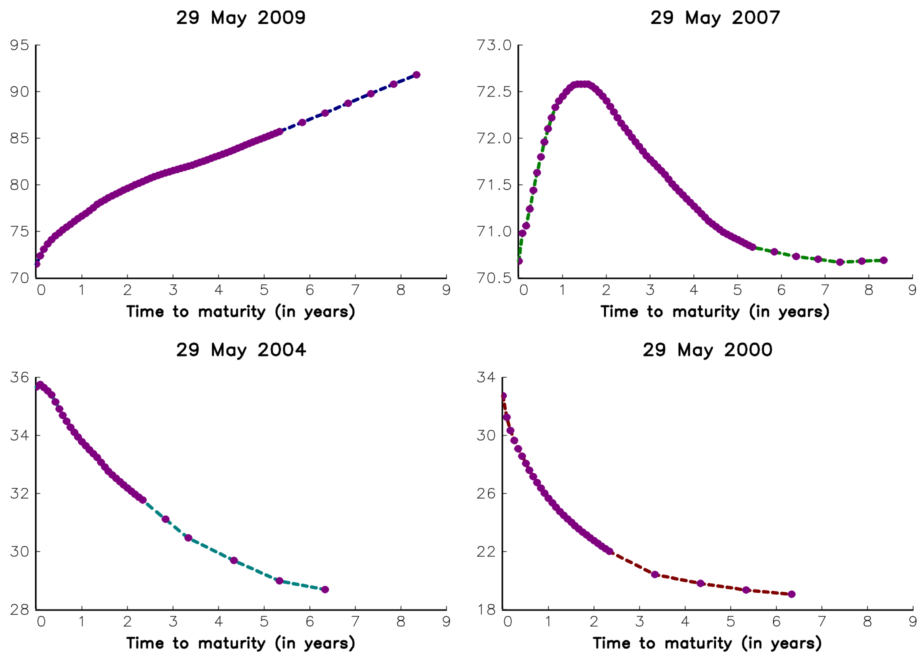
Figure 12: Contango and backwardation movements



As for fixed-income instruments, there are several approaches to implementing a commodity carry strategy. The first consists in selecting the long and short exposure from a set of commodities. In this case, the (normalized) carry is equal to:

$$\mathcal{C}_{i,t}(T) = \frac{S_{i,t} - F_{i,t}(T)}{(T-t)\,F_{i,t}(T)}$$

Figure 13: Term structure of crude oil futures contracts



We obtain the following rule:

$$\mathcal{C}_{i,t}(T) > \mathcal{C}_{j,t}(T) \Longrightarrow i \succ j$$

This means that we prefer commodity i over commodity j because it has higher carry. This is why long positions generally concern commodities in backwardation (and, conversely, short positions generally concern commodities in contango). The second approach examines the slope of two futures contracts whose maturities are T_1 and T_2 with $T_2 > T_1$. In this case, the carry is equal to the "*roll down*" between the two futures contracts:

$$\mathcal{C}_{i,t}(T_2 \mid T_1) = \frac{F_{i,t}(T_1) - F_{i,t}(T_2)}{(T_2 - T_1) F_{i,t}(T_2)}$$

For a given commodity i, we can then compare the different values of roll down and choose the pair of maturities (T_1, T_2) according to the following rule:

$$\mathcal{C}_{i,t}(T_2 \mid T_1) > \mathcal{C}_{i,t}(T_2' \mid T_1') \Longrightarrow (T_1, T_2) \succ (T_1', T_2')$$

Finally, the third carry strategy compares the slope of two commodity term structures:

$$\mathcal{C}_{i,t}(T_2 \mid T_1) > \mathcal{C}_{j,t}(T_2 \mid T_1) \Longrightarrow i \succ j$$

Again, we distinguish between the three carry strategies by calling them "*Carry – Forward Rate Bias (FRB)*", "*Carry – Term Structure Slope (TSS)*" and "*Carry – Cross Term Structures (CTS)*".

The risk of the carry strategy The carry strategy is based on the forward rate bias puzzle, which states that the forward price is a biased estimator of the spot price. This empirical result contradicts the financial theory of rational expectations. The carry strategy is then related to the literature on behavioral finance. The question of the risk premium thus becomes an issue. However, an historical analysis shows that carry strategies are extremely risky in periods of market distress and liquidity risk. This explains why they exhibit tail risks and may suffer from large drawdowns.

3.1.2 Liquidity

The liquidity factor is similar to the size factor for equities. The idea is gain risk exposure as a result of the illiquidity of the assets. Even if the liquidity factor is well-defined in academia (Pàstor and Stambaugh, 2003), it seems rather vague in the industry, because liquidity measures are difficult to define for illiquid assets. The liquidity factor therefore encompasses different types of strategy and different schemes of implementation. This is why the liquidity premium is mainly captured by hedge funds.

In the equity asset class, one of the most popular illiquidity measures is the Amihud ratio (Amihud, 2002) defined as follows:

$$\mathcal{L}_{i,t} = \frac{1}{m} \sum_{h=1}^{m} \frac{|R_{i,t-h}|}{\mathrm{DTV}_{i,t-h}}$$

This is the average ratio of the daily absolute return to the dollar trading volume over a given period[25]. To capture the liquidity premium, the investor will prefer asset i to asset j if the illiquidity measure of asset i is higher:

$$\mathcal{L}_{i,t} \geq \mathcal{L}_{j,t} \Longrightarrow i \succ j$$

In the other asset classes, liquidity strategies are based on less technical measures and consist in market timing strategies. For instance, in the fixed-income universe, they may result in an arbitrage strategy of going long off-the-run bonds and short on-the-run bonds. Indeed, bond indices generally rebalance on a fixed day (e.g. the last trading day of the month) in order to include newly issued bonds and exclude bonds that have rolled out. Consequently, index fund managers must buy these new bonds at the rebalancing date, which may create price pressures. This pattern is known as the turn-of-the-month effect.

This pattern may also be observed in commodity futures markets. Passive funds and structured products based on the S&P GSCI or BCOM indices can reach a large size in terms of managed assets. One of the main characteristics of these two indices is that the futues contracts are rolled over a similar time window. For instance, the S&P GSCI index rolls over five days between the fifth and ninth business day, whereas the BCOM index rolls between the sixth and tenth business day. As a result, liquidity pressures appear around the turn of the month because large investment flows simultaneously go into selling front month futures contracts and buying next out futures contracts.

3.1.3 Momentum

The momentum risk factor has been extensively documented both for equities (Jegadeesh and Titman, 1993; Carhart, 1997) and commodities (Erb and Harvey, 2006; Miffre and Rallis, 2007). Moskowitz *at al.* (2012) and Asness *et al.* (2013) discuss also the presence of

[25] It is generally equal to one year.

momentum in other asset classes, for instance in currencies and fixed-income instruments. The momentum strategy is well-known in investment management and has been used by hedge funds and CTAs for many years (Lempérière *et al.*, 2014b). It corresponds to the trend-following strategy and is called "*time-series momentum*" by Moskowitz *at al.* (2012). Nevertheless, a variant of this strategy was proposed a long time ago by Carhart (1997) for the purposes of analyzing the return on equity portfolios. This second strategy is known as "*cross-section momentum*".

The momentum of asset i at time t corresponds to its past return:

$$\mathcal{M}_{i,t} = \frac{P_{i,t} - P_{i,t-h}}{P_{i,t}}$$

where $P_{i,t}$ is the asset price and h is the momentum period. Using cross-section momentum, we have:

$$\mathcal{M}_{i,t} > \mathcal{M}_{j,t} \Longrightarrow i \succ j$$

This means that the portfolio is long on assets that present a momentum higher than the other assets. For instance, the WML risk factor identified by Carhart (1997) is a long/short portfolio with the same number of constituents for the two legs. This momentum strategy differs from the time-series momentum strategy, which is defined by:

$$\left\{ \begin{array}{l} \mathcal{M}_{i,t} > 0 \Longrightarrow i \succ 0 \\ \mathcal{M}_{i,t} < 0 \Longrightarrow i \prec 0 \end{array} \right.$$

In this case, the portfolio is long on asset i if it has positive momentum (and, conversely, short on asset i if it has negative momentum). This strategy is also called the "*trend continuation*", because it assumes that the past trend is a predictor of the future trend. This contrasts with the cross-section momentum strategy, where assets with negative trends can compose long exposure[26].

Calibrating a momentum strategy is not an easy task. One-year past return is the standard measure widely used by academics, who find persistence in returns for one to 18 months and reversal in returns for shorter and longer horizons. Professionals use more sophisticated techniques (Bruder *et al.*, 2014). Moreover, trading execution and transaction costs are key factors when implementing such a strategy. This explains why there may be a discrepancy between academic risk factors and trend-following strategies.

3.1.4 Reversal

The reversal strategy may be defined as the opposite of the momentum strategy. It is also known as the mean-reverting strategy. These strategies use short-term trends (for instance shorter than one month) or very long-term trends. There are several ways to construct the reversal signal. It can be estimated using autocorrelation functions, differences in variance of time-aggregated processes or moving average crossovers.

Let $P_{i,t}$ be the price of asset i at time t. We define the one-period return as follows:

$$R_{i,t} = \ln P_{i,t} - \ln P_{i,t-1}$$

[26] Indeed, the cross-section momentum verifies that:

$$\mathcal{M}_{i,t} > \bar{\mathcal{M}}_t \Longrightarrow i \succ 0$$

where $\bar{\mathcal{M}}_t = n^{-1} \sum_{j=1}^{n} \mathcal{M}_{j,t}$ is the average momentum of the asset universe. An asset with negative momentum higher than the average is therefore a candidate for the long leg of the portfolio.

We note $\rho_i(h) = \rho(R_{i,t}, R_{i,t-h})$ the autocorrelation function of returns. Asset i exhibits a mean-reverting pattern if the short-term autocorrelation $\rho_i(1)$ is negative. In this case, the short-term reversal is defined by the product of the autocorrelation and the current return:

$$\mathcal{R}_{i,t} = \rho_i(1) \times R_{i,t}$$

The short-term reversal strategy is then defined by the following rule:

$$\mathcal{R}_{i,t} \geq \mathcal{R}_{j,t} \Longrightarrow i \succ j$$

In particular, if $\mathcal{R}_{i,t}$ is positive, meaning that the current return $R_{i,t}$ is negative, we should buy the asset, because a negative return is followed by a positive return on average.

This short-term reversal has a significant impact on the dynamic of the return variance. Let us assume that the one-period asset return follows an AR(1) process:

$$R_{i,t} = \rho R_{i,t-1} + \varepsilon_t$$

where $|\rho| < 1$, $\varepsilon_t \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right)$ and $\operatorname{cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ for $j \geq 1$. Let $\mathrm{RV}(h)$ be the annualized realized variance of the h-period asset return $R_{i,t}(h) = \ln P_{i,t} - \ln P_{i,t-h}$. Using the results in Appendix C.5 concerning the variance of time-aggregated processes, we show that:

$$\mathbb{E}[\mathrm{RV}(h)] = \phi(h)\,\mathbb{E}[\mathrm{RV}(1)]$$

with:

$$\phi(h) = 1 + 2\rho\frac{1-\rho^{h-1}}{1-\rho} - 2\sum_{j=1}^{h-1}\frac{j}{h}\rho^j$$

We notice that:

$$\lim_{h\to\infty}\mathbb{E}[\mathrm{RV}(h)] = \left(1 + \frac{2\rho}{1-\rho}\right)\mathbb{E}[\mathrm{RV}(1)]$$

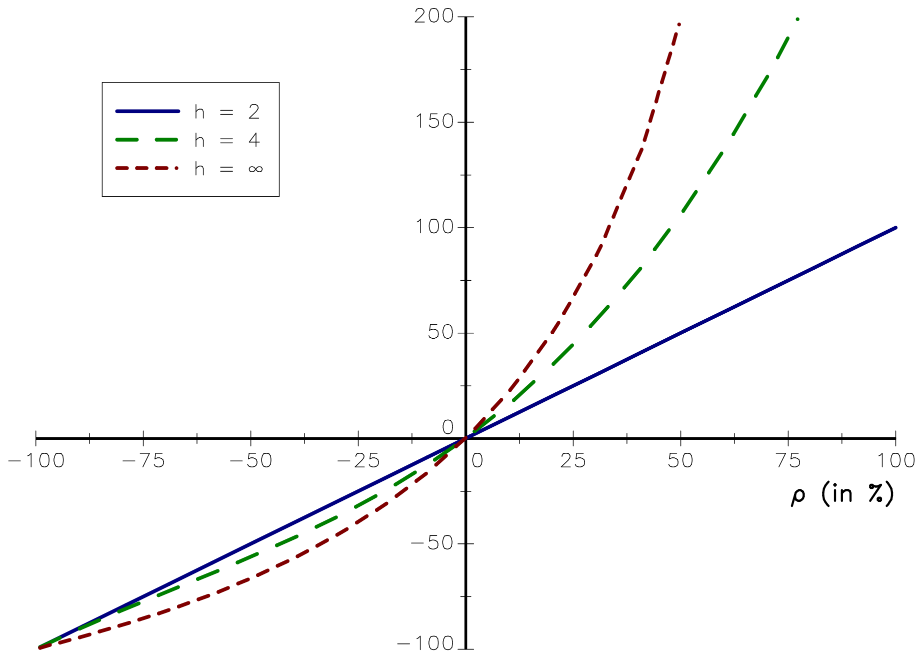
When the autocorrelation is negative, this implies that the long-term frequency variance is lower than the short-term frequency variance. More generally, we have:

$$\begin{cases} \mathbb{E}[\mathrm{RV}(h)] < \mathbb{E}[\mathrm{RV}(1)] & \text{if } \rho < 0 \\ \mathbb{E}[\mathrm{RV}(h)] \geq \mathbb{E}[\mathrm{RV}(1)] & \text{otherwise} \end{cases}$$

In Figure 14, we report the variance ratio $(\mathrm{RV}(h) - \mathrm{RV}(1)) / \mathrm{RV}(1)$ for several values of ρ and h. The spread between daily/weekly and weekly/monthly variance swaps therefore depends on the autocorrelation of daily returns. In this case, the reversal strategy consists in being long on the daily/weekly variance swaps and short on the weekly/monthly variance swaps.

Short-term return reversal in the stock market has been a well-established phenomenon for more than 40 years (De Bondt and Thaler, 1985). In particular, the pattern of negative autocorrelation from one-day to one-month was well documented at the beginning of the 1990s (Jegadeesh, 1990; Lehmann, 1990; Lo and MacKinlay, 1990). Generally, two possible arguments are given to explain short-term reversal profits. The sentiment-based explanation suggests that short-term market prices may reflect an overreaction on the part of investors to information or fads, whereas the liquidity-based explanation suggests that reversal strategy profits mainly derive from positions in small and illiquid stocks (Avramov *et al.*, 2006).

Figure 14: Variance ratio $(\mathrm{RV}(h) - \mathrm{RV}(1)) / \mathrm{RV}(1)$ (in %)



Remark 2 *The long-term return reversal is defined by the difference between short-period and long-run average prices:*

$$\mathcal{R}_{i,t} = \bar{P}_{i,t}^{ST} - \bar{P}_{i,t}^{LT}$$

Typically, $\bar{P}_{i,t}^{ST}$ is the average price over the last year and $\bar{P}_{i,t}^{LT}$ is the average price over the last five years. The long-term return reversal strategy follows the same rule as the short-term reversal strategy. This strategy is described as a value strategy because the long-run average price can be viewed as an estimate of the fundamental price in some asset classes. This is why we reserve the term "reversal" to the short-term strategy.

3.1.5 Value

The value equity factor was studied in great depth by Fama and French (1993, 1998, 2012). This strategy goes long under-valuated stocks and short over-valuated stocks. As seen previously, it is a typical mean-reverting strategy. One of the issues associated with this strategy is precisely defining the value measure. Whereas the most common metric is the book-to-market ratio, other measures may be implemented and combined to define the value signal (earnings yield, free cash flow yield, etc.). This value signal may also use lagged, contemporary or estimated data, implying that the behavior of the value equity factor depends heavily on the implementation scheme (Asness and Frazzini, 2013).

The value strategy has been extended to other asset classes by academic research and index providers. Let $P_{i,t}$ and $\bar{P}_i$ be the market price and the fundamental price of the asset i. The value of asset i measures the valuation difference between these two prices:

$$\mathcal{V}_{i,t} = \frac{P_{i,t} - \bar{P}_i}{\bar{P}_i}$$

As in equities, there are several ways to define the fair price $\bar{P}_{i,t}$. We generally encounter two main approaches. The economic approach uses theoretical models to define the long-run equilibrium price of asset i. For instance, we can use the theory of purchasing power parity (PPP) to define the fair value of exchange rates. A related measure is the real effective exchange rate (REER). The second approach consists in defining the long-run equilibrium price using statistical methods. For instance, we can approximate the value by using the 5-year spot return or a moving average crossover. The underlying idea is to compare the current level to the long-term average of spot prices. This is the approach used by Asness *et al.* (2013) for commodities. For fixed-income instruments, the value factor is more anecdotal[27].

3.1.6 Volatility

As explained by Brière *et al.* (2010), a structural long position with regard to volatility is a natural way to protect long-term portfolios in distressed periods. This is why academics and professionals have extensively investigated investments in the VIX index in a strategic asset allocation framework. They have showed that incorporating the volatility asset class improved the risk/return profile of Markowitz portfolios, because they benefit from diversification as a result of the negative correlation between volatility and return. However, all these studies suffer from a significant drawback. They largely underestimate the transaction and replicating costs associated with a long volatility exposure. In practice, long-term institutional investors do not adopt this strategy, because it is too costly.

In fact, the traditional volatility risk premium corresponds to a portfolio that captures the spread between implied volatility and realized volatility. This volatility risk premium is related to the robustness formula of the Black-Scholes model. On the basis of certain assumptions, El Karoui *et al.* (1998) showed that the P&L of selling and delta-hedging a convex payoff is equal to:

$$\Pi = \frac{1}{2}\int_0^T e^{r(T-t)} S_t^2 \Gamma_t \left(\Sigma_t^2 - \sigma_t^2\right) \, \mathrm{d}t$$

where S_t is the price of the underlying asset, Γ_t is the gamma coefficient, Σ_t is the implied volatility and σ_t is the realized volatility. The seller of the option should then price the option with an implied volatility Σ_t that is larger than the realized volatility σ_t in order to obtain a positive P&L. We deducted this because of the asymmetric risk profile between the seller and the buyer, hedging demand imbalances, and liquidity preferences. This volatility risk premium can then be captured by being long on implied volatility and short on realized volatility. This strategy can be implemented in different ways using call/put options, straddle/strangle derivatives, variance swaps or VIX futures contracts. It is widely used with equity indices, but we may also find it in the fixed-income and foreign exchange asset classes. In the case of commodities, it mainly affects options on energy futures contracts.

The aforementioned strategy is also called the volatility/carry strategy and is sometimes grouped together with the carry risk premium. Another strategy concerns the term structure of VIX futures contracts. Like the carry/term structure slope strategy, the volatility/term structure slope strategy aims to capture the net roll-down effect, which is generally located around the 2M/3M maturity.

[27] When this factor is defined, it is more closely aligned with a scoring system designed to mimic the bond picking process (Correia *et al.*, 2012).

3.1.7 Equity-specific risk factors

Besides the aforementioned risk factor strategies found in the various asset classes, there are some risk premia candidates that are specific to equity asset classes. Short definitions of each risk factor are provided below, but the reader can find a more detailed description in Cazalet and Roncalli (2014).

Event The category called "*event*" covers several idiosyncratic risk strategies. Typical examples are merger arbitrage or convertible arbitrage. Because these strategies are exposed to distress risk, they are partially related to the liquidity risk factor. The event category also includes the buyback strategy, which invests in companies that announce stock repurchases.

Growth The growth strategy consists in selecting stocks of companies that are growing substantially faster than others. Contrary to popular belief, this is not the same as the anti-value strategy.

Low volatility The low volatility anomaly consists in building a portfolio with exposure to low volatility (or low beta) stocks. The recent success of minimum variance portfolios explains the development of this factor, although it was described by Fisher Black in 1972. Frazzini and Pedersen (2014) extended Black's original work to other asset classes. Nevertheless, the implementation of the low volatility factor only affected equities until now.

Quality Like the low volatility factor, the quality strategy is another market anomaly that cannot be explained by a risk pattern. It corresponds to a portfolio invested in quality stocks without any reference to the market price. Quality measures include the equity-to-debt, return-on-equity and income/sales ratios. This equity factor is more recent than the Fama-French risk factors (Piotroski, 2000).

Size The size factor has been implemented by pension funds investing in equities for many years. The underlying idea is that small stocks have a natural excess return with respect to large stocks and this excess return may be explained by a liquidity premium. While the size factor was popularized in a long/short format by Fama and French (1993), it is typically implemented in long-only portfolios by institutional investors[28].

3.2 Statistical analysis

When academics or professionals want to identify alternative risk premia, they generally perform a backtest and then proceed to a statistical analysis. In this paper, we approach the study of these risk factor strategies from another angle. We compile a database using existing indices, which are sponsored and calculated by asset managers, banks and index providers. In Appendix A, we provide a description of this database. It contains about 2 000 products sold as alternative risk premia. Each product is tagged according to the previous classification, which is summarized in Figure 16. Through an analysis of all these indices, it becomes apparent that the performance of many products does not reflect the returns of an alternative risk premium. We encounter three types of situation:

1. Some of them use the term "*alternative risk premia*" for marketing purposes, but the goal of the strategy is not to capture a risk premium. Instead, it would be better classified as a proprietary investment model. In such cases, it is difficult to find another index provider that proposes something similar.

[28] Until now, the size factor was exclusively related to equities. Houweling and van Zundert (2015) suggested that the size factor may also be present in the credit market, but there is no other study to confirm this.

2. Others combine different risk premia, for instance momentum and carry.

3. Finally, some products are optimized to produce in-the-sample high Sharpe ratios, which are not consistent with those observed since live dates. For instance, they may include stop-loss and take-profits mechanisms, consider Markov-switching regimes or use dynamic allocation.

After excluding the products that fall into these three categories, we obtain a clean database of 1 382 products. After this due diligence, the second step consists in estimating the generic performance of each risk premium by averaging the returns of the various indices that replicate the risk premium. However, we notice that the performance between the indices can be highly heterogeneous. That is why the generic performance is calculated using the selection procedure described in Appendix B. An example of the equities/volatility/carry/US risk premium is provided in Figure 15. There are 14 excess return indices after the first step, but the generic performance is calculated by using only the five indices that are selected after the second step.

Summary 12 *Performing due diligence on alternative risk premia products is a difficult task. At first sight, it appears that there are many products that replicate a given risk premium. In practice, few number of products are relevant, because most of them do not reflect a pure exposure to the risk premium. For example, in the case of the equity/volatility/carry/US risk premium,* 30 *products in our database are presented as investment vehicles replicating this risk premium. After analyzing the description of the strategy in detail,* 16 *products are eliminated. Finally, only five products are selected by using the quantitative procedure, which detects average behavior.*

Figure 15: An example of building a generic cumulative performance

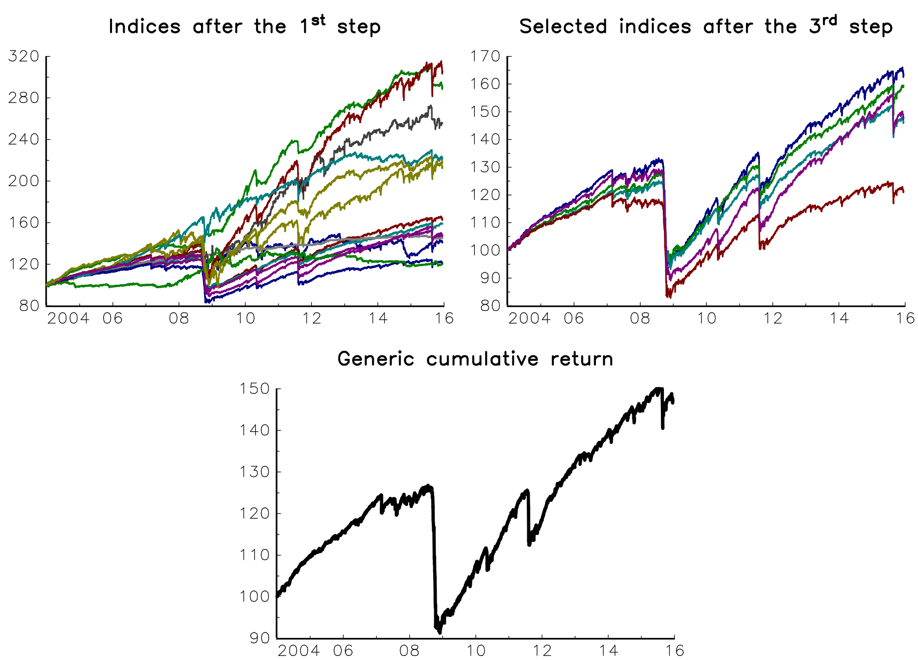


Figure 16: Classification of the ARP database

Risk Factor	Equities	Rates	Credit	Currencies	Commodities
Carry	Dividend Futures High Dividend Yield	FRB TSS CTS	FRB	FRB	FRB TSS CTS
Liquidity	Amihud liquidity	Turn-of-the-month	Turn-of-the-month		Turn-of-the-month
Momentum	Cross-section Time-series	Cross-section Time-series	Time-Series	Cross-section Time-series	Cross-section Time-series
Reversal	Time-series Variance	Time-series		Time-series	Time-series
Value	Value	Value	Value	PPP Economic model	Value
Volatility	Carry Term structure	Carry Term structure		Carry	Carry
Event	Buyback Merger arbitrage				
Growth	Growth				
Low volatility	Low volatility				
Quality	Quality				
Size	Size				

Table 3: The most frequent ARP indices

Asset class	Risk premium		Format	Region	Currency	n
Equities	Value		Long	US	USD	27
Commodities	Carry	FRB	Long/short	DM + EM	USD	22
Equities	Carry	HDY	Long	US	USD	20
Equities	Low volatility		Long	Europe	EUR	19
Equities	Value		Long	Europe	EUR	18
Commodities	Momentum	Cross-section	Long/short	DM + EM	USD	18
Rates	Volatility	Carry	Long/short	US	USD	18
Equities	Low volatility		Long	US	USD	16
Currencies	Carry	FRB	Long/short	DM	USD	16
Currencies	Carry	FRB	Long/short	DM + EM	USD	16
Rates	Volatility	Carry	Long/short	Europe	EUR	16
Commodities	Momentum	Time-series	Long/short	DM + EM	USD	14
Equities	Volatility	Carry	Long/short	US	USD	14
Commodities	Liquidity		Long/short	DM + EM	USD	12
Equities	Momentum	Cross-section	Long	US	USD	12
Commodities	Carry	TSS	Long/short	DM + EM	USD	11
Equities	Carry	HDY	Long	Europe	EUR	11
Equities	Growth		Long	US	USD	11
Rates	Volatility	Carry	Long/short	DM	USD	11
Equities	Carry	HDY	Long	EM	USD	8
Equities	Quality		Long	Europe	EUR	8
Equities	Value		Long	Japan	JPY	8
Equities	Carry	HDY	Long	Asia Pacific	USD	7
Equities	Low volatility		Long/short	Europe	EUR	7
Equities	Quality		Long/short	Europe	EUR	7
Currencies	Momentum	Time-series	Long/short	DM	USD	7
Rates	Momentum	Time-series	Long/short	DM	USD	7
Currencies	Carry	FRB	Long/short	EM	USD	6
Rates	Momentum	Time-series	Long/short	US	USD	6
Rates	Volatility	Carry	Long/short	Japan	JPY	6

In Table 3, we report the most common risk premia[29]. To this end, we count the number of different indices in the database that are selected after the due diligence step for a given region (DM, EM, US, Europe, Japan, Asia Pacific, etc.), a given currency (USD, EUR, JPY, etc.) and a given format (long, long/short, etc.). For instance, there are 27 indices that harvest the US value equity premium in a long-only format and are denominated in USD. We notice that the most popular ARPs are common equity risk factors. However, some risk premia of other asset classes are frequently encountered, such as commodities/carry, commodities/momentum, currencies/carry, rates/volatility, currencies/momentum and rates/momentum. In terms of importance, we obtain the following ranking:

1. Equities: value, carry (HDY), low volatility, volatility (carry), momentum (cross-section), quality, growth, size, event (merger arbitrage), reversal (time-series);

2. Rates: volatility (carry), momentum (time-series), carry (TSS and FRB);

3. Credit: momentum (time-series).

[29] This table is provided for reference only. For the long-only format, we only consider net return indices, whereas we use excess return indices for the long/short format.

4. Currencies: carry (FRB), momentum (time-series), value (PPP);

5. Commodities: carry (FRB and TSS), momentum (cross-section and time-series), liquidity;

We notice that most equity ARP indices are in a long-only format. Long/short indices mainly concern volatility and event risk premia. Curiously, the size factor is not very often present, because it is generally replicated using a CW portfolio[30]. Carry and momentum risk premia are present in all asset classes (commodity, rates, currencies), but this is not the case for the value risk premium.

Summary 13 *Our ARP database shows that carry and momentum premia can be found in all asset classes, but this is not the case for other risk premia. For instance, the value premium is mainly associated with the equity asset class; moreover, we find the volatility risk premium in the equity and fixed-income asset classes, but not really in the other asset classes.*

3.2.1 Generic performance of alternative risk premia

In the following paragraphs, we report the generic cumulative return of ARPs for the different asset classes, which are obtained by setting $\mathbf{R}^2_{\min} = 70\%$ and $n_{\min} = 5$. All the backtests are in long/short format, except for equities.

The case of equity risk premia is the most complex. First, two formats (long-only and long/short) exist. However, the number of long/short indices is generally low, meaning that the constraint $n \geq n_{\min}$ may not be satisfied. For this reason, we prefer to deal with long-only indices. The exceptions are the volatility, event and reversal risk premia, which are exclusively available in long/short format[31] (Figures 17–19). Finally, we obtain 29 long-only risk premia and seven long/short risk premia. The "*traditional*" ARPs (carry, value, momentum, growth, low volatility and quality) can be found in the US, Europe, Japan and Asia Pacific. Some risk factors are also multi-regional (in particular, carry and value). The long/short indices (volatility, event and reversal) are mainly associated with the US market.

In the case of rates (Figure 20), we obtain 10 alternative risk premia (carry, momentum and volatility), which mainly affect the US and European regions. The number of Japanese indices is too small to satisfy the constraint $n \geq n_{\min}$, and only the volatility carry risk premium can be calculated. We obtain eight currency ARP backtests, which are associated with the carry, momentum and value risk factors (Figure 21). These indices mainly focus on two regions: developed markets and global markets (including emerging markets). However, we notice that the value premium is based on developed markets. For the commodity asset class (Figure 22), we have identified five alternative risk premia: two carry strategies (FRB and TSS), two momentum strategies (cross-section and time-series) and the liquidity strategy (turn-of-the-month effect). These products are global and include energy, metal and agriculture commodities.

Remark 3 *On page 114, we show the generic performance for credit risk premia. This is for reference only, as the constraint $n \geq n_{\min}$ is never satisfied*[32].

[30] As a reminder, we only consider non cap-weighted indices in the database.
[31] The US merger arbitrage risk factor may also be found in a long-only format.
[32] The number of indices per ARP is between one and four. The latter pertains to the momentum risk factor in European credit.

Figure 17: ARP generic indices (equities)

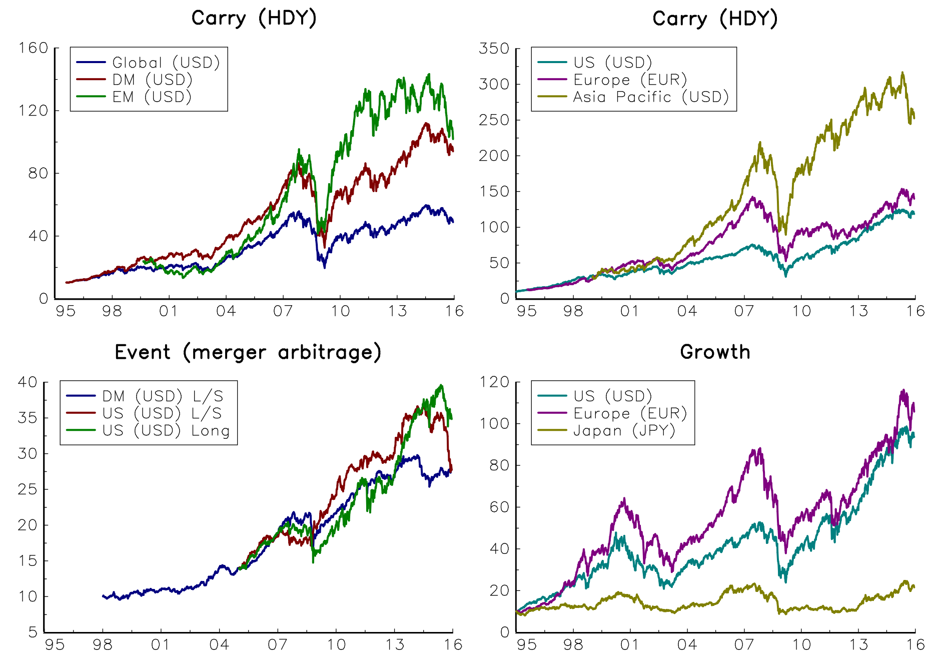


Figure 18: ARP generic indices (equities)

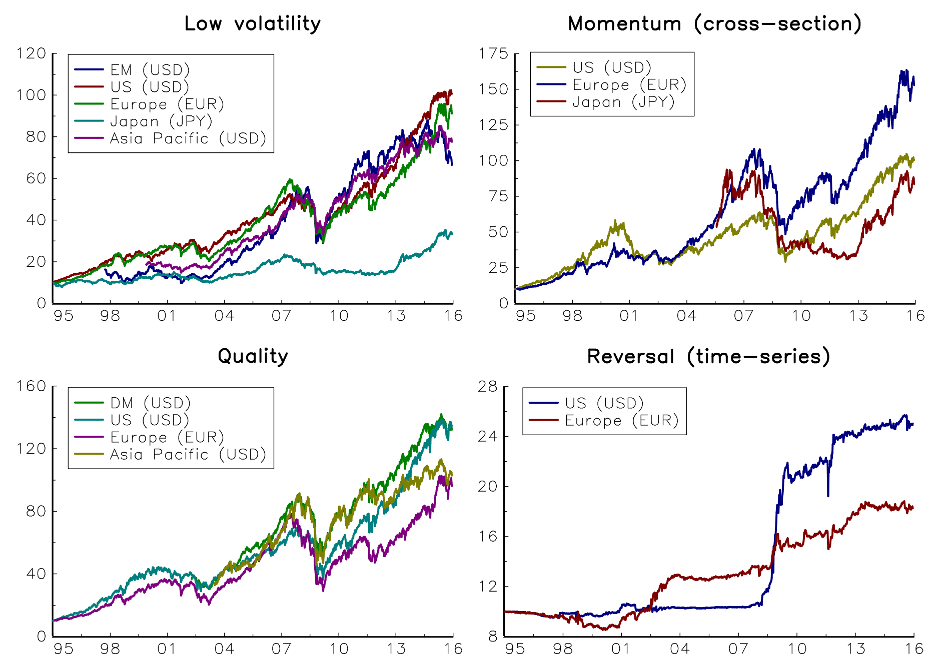


Figure 19: ARP generic indices (equities)

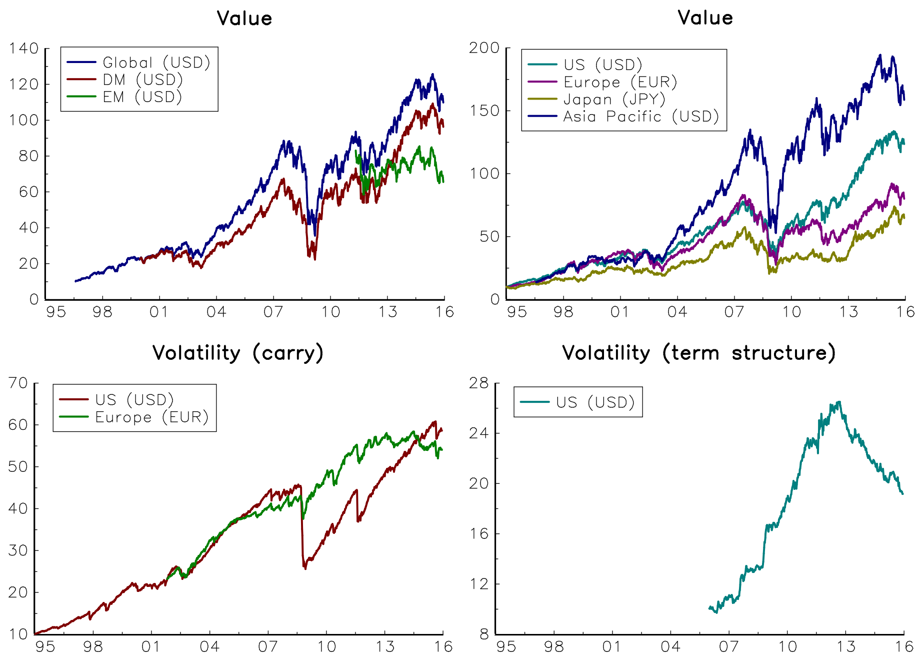


Figure 20: ARP generic indices (rates)

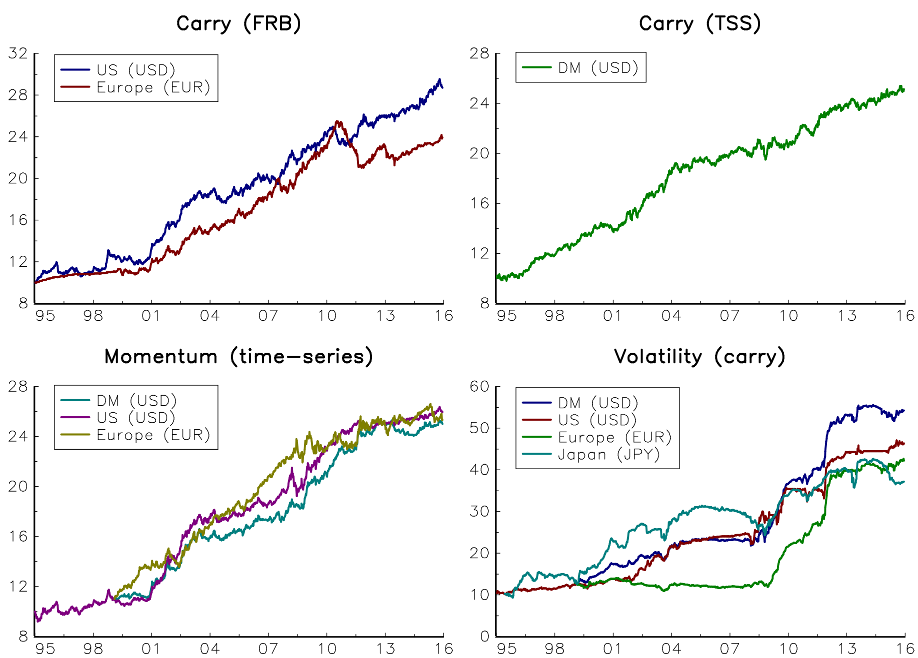


Figure 21: ARP generic indices (currencies)

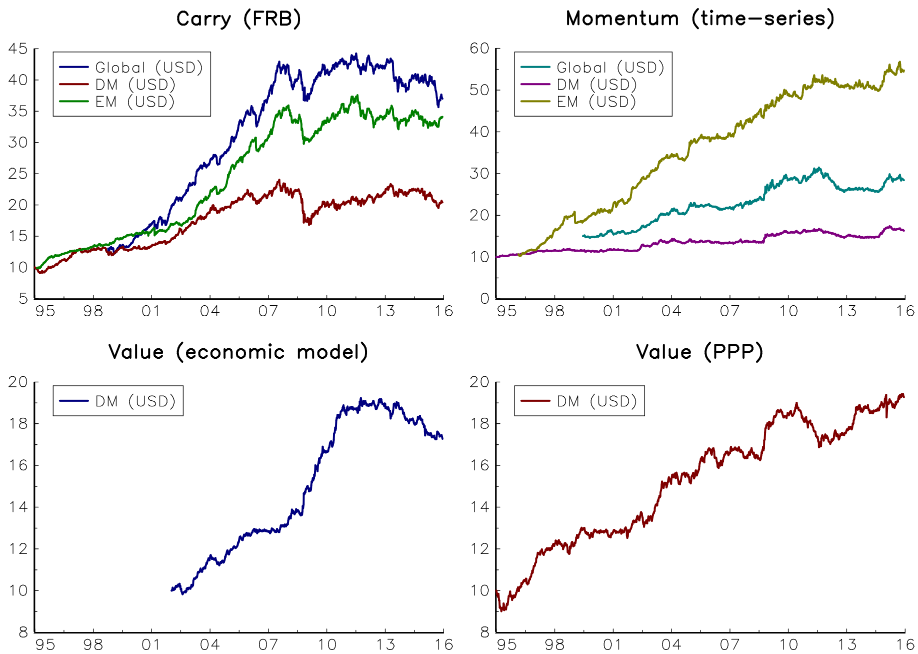
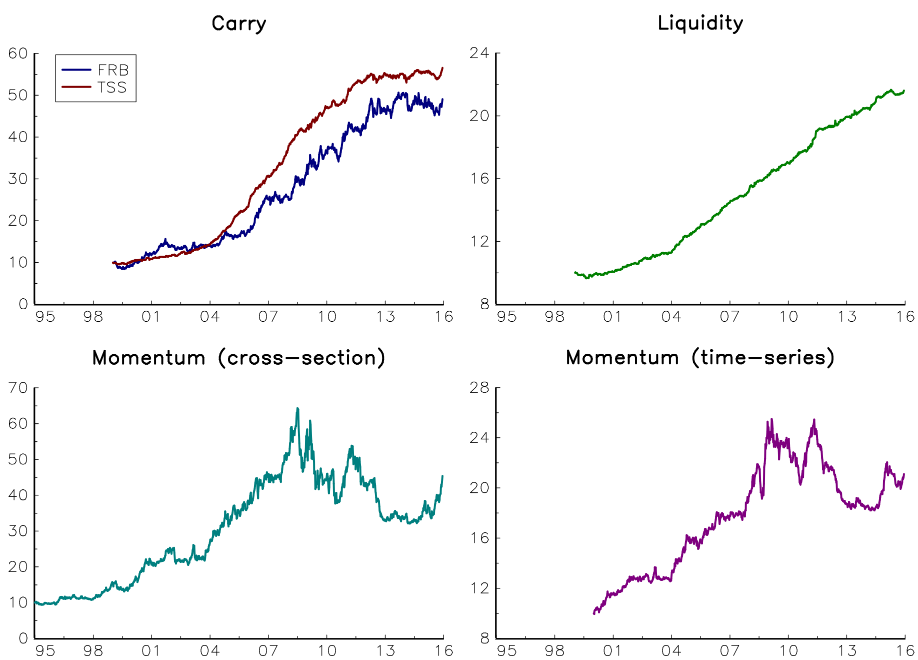


Figure 22: ARP generic indices (commodities)



3.2.2 Risk/return analysis of ARP generic indices

In Tables 4 and 5, we report the Sharpe ratio SR, the standard deviation of the Sharpe ratio $\hat{\sigma}$ (SR), the normalized drawdown $\mathcal{DD}^{\star}$ and the skewness coefficient γ_1 of traditional and alternative risk premia for the period from January 2000 to December 2015. The normalized drawdown corresponds to the maximum drawdown divided by the volatility of positive returns. If we consider traditional risk premia, the Sharpe ratio is low for equities[33] and high for sovereign and corporate bonds[34]. We explain this result with reference to the period, which was characterized by falling interest rates in developed markets. On average, the maximum drawdown is 4.3 times greater than the volatility of positive returns. Except for the MSCI Europe index, we also observe a negative skewness, but the latter is generally low.

Table 4: Statistics of traditional risk premia

Name	SR	$\mathcal{DD}^{\star}$	γ_1
MSCI ACWI index (USD)	0.10	5.02	-0.21
MSCI WORLD index (USD)	0.10	4.89	-0.20
MSCI EM index (USD)	0.18	4.93	-0.31
MSCI United States index (USD)	0.11	3.71	-0.01
MSCI Europe index (EUR)	0.05	4.01	0.03
MSCI Japan index (JPY)	0.04	3.97	-0.24
MSCI AC Asia Pacific index (USD)	0.15	4.55	-0.29
Barclays Global Agg Govt index (USD)	1.15	2.97	-0.21
Barclays US Agg Govt index (USD)	0.68	2.17	-0.13
Barclays Euro Treasury Bond index (EUR)	0.88	2.73	-0.06
Citi Japanese Govt Bond index (JPY)	0.81	3.15	-0.10
Barclays Global Agg Corporate index (USD)	1.00	5.47	-0.39
Barclays US Agg Corporate index (USD)	0.75	4.77	-0.24
Barclays Pan European Agg Corporate index (EUR)	0.94	5.61	-0.54
Bloomberg JP Morgan Asia Dollar index (USD)	-0.57	5.40	-0.16
Bloomberg Dollar Spot index (USD)	-0.26	7.40	-0.02
Bloomberg Commodity index (USD)	-0.07	6.17	-0.18

If we consider the ARP generic indices, we observe some differences with respect to the results obtained from traditional long-only indices. On average, the Sharpe ratio is higher for equities[35], whereas it is in line with long-only benchmarks for rates. In the case of currencies and commodities, we also obtain very high Sharpe ratios. For each ARP, we indicate the standard deviation $\hat{\sigma}$ (SR), calculated using the different indices that compose the ARP generic index. This statistic is given for reference only, because it depends on the number of relevant indices used to estimate the generic index. With regards to normalized drawdown, some alternative risk premia present extreme behaviors that we do not observe when we consider traditional risk premia. For instance, this is the case for the equities/volatility/carry/US risk factor, whose drawdown is 12.74 times greater than the volatility of positive returns. Conversely, the normalized drawdown is particularly low for equities/reversal, rates/carry/TSS, commodities/carry/TSS and commodities/liquidity. If we analyze the skewness coefficient, some ARPs have a positive or low value: equities/event/merger arbitrage, equities/quality and equities/reversal. The most skewed risk

[33]With a range between 0.04 and 0.18.

[34]The Sharpe ratio is equal to 1.15 for global government bonds and 1.00 for global corporate bonds.

[35]The mean across regions is equal to 0.77 for volatility/carry, 0.62 for event/merger arbitrage 0.47 for low volatility, 0.37 for carry/HDY and 0.35 for quality.

premia are equities/volatility/carry, rates/volatility/carry, currencies/carry/FRB and currencies/value/PPP.

Table 5: Statistics of ARP generic backtests

i	Name	SR	$\hat{\sigma}$ (SR)	$\mathcal{DD}^{\star}$	γ_1
1	equities/carry/HDY/Global (USD)	0.20	0.15	5.30	-0.18
2	equities/carry/HDY/DM (USD)	0.38	0.10	5.53	-0.29
3	equities/carry/HDY/EM (USD)	0.41	0.27	5.06	-0.49
4	equities/carry/HDY/US (USD)	0.34	0.09	4.04	-0.07
5	equities/carry/HDY/Europe (EUR)	0.30	0.18	4.80	-0.03
6	equities/carry/HDY/Asia Pacific (USD)	0.58	0.10	5.49	-0.71
7	equities/event/merger arbitrage/DM (USD)	0.71	0.21	3.71	0.82
8	equities/event/merger arbitrage/US (USD)	0.68	0.09	3.94	0.26
9	equities/event/merger arbitrage/long/US (USD)	0.48	0.09	2.64	0.12
10	equities/growth/US (USD)	0.14	0.09	3.45	-0.02
11	equities/growth/Europe (EUR)	0.12	0.13	4.07	-0.01
12	equities/growth/Japan (JPY)	0.05	0.10	4.03	-0.12
13	equities/low volatility/EM (USD)	0.48	0.19	5.17	-0.47
14	equities/low volatility/US (USD)	0.48	0.11	3.95	-0.02
15	equities/low volatility/Europe (EUR)	0.45	0.08	5.14	-0.16
16	equities/low volatility/Japan (JPY)	0.39	0.01	4.49	-0.85
17	equities/low volatility/Asia Pacific (USD)	0.54	0.01	4.65	-0.54
18	equities/momentum/cross-section/US (USD)	0.09	0.11	3.61	-0.11
19	equities/momentum/cross-section/Europe (EUR)	0.41	0.08	4.27	-0.23
20	equities/momentum/cross-section/Japan (JPY)	0.17	0.08	4.51	-0.56
21	equities/quality/DM (USD)	0.53	0.00	4.12	0.08
22	equities/quality/US (USD)	0.27	0.14	3.36	0.08
23	equities/quality/Europe (EUR)	0.26	0.10	4.59	-0.07
24	equities/quality/Asia Pacific (USD)	0.36	0.72	3.42	-0.04
25	equities/reversal/time-series/US (USD)	0.54	0.15	1.68	3.06
26	equities/reversal/time-series/Europe (EUR)	0.52	0.03	1.46	1.86
27	equities/value/Global (USD)	0.45	0.20	4.77	-0.19
28	equities/value/DM (USD)	0.35	0.02	4.64	-0.12
29	equities/value/EM (USD)	-0.35	0.36	2.93	-0.34
30	equities/value/US (USD)	0.34	0.12	3.88	-0.16
31	equities/value/Europe (EUR)	0.16	0.12	4.39	0.03
32	equities/value/Japan (JPY)	0.29	0.08	3.85	-0.04
33	equities/value/Asia Pacific (USD)	0.45	0.17	4.89	-0.21
34	equities/volatility/carry/US (USD)	0.64	0.19	12.74	-7.44
35	equities/volatility/carry/Europe (EUR)	0.90	0.31	3.94	-1.96
36	equities/volatility/term structure/US (USD)	0.78	0.01	4.23	0.16
37	rates/carry/FRB/US (USD)	1.04	0.01	2.15	-0.15
38	rates/carry/FRB/Europe (EUR)	0.99	0.00	5.41	-0.41
39	rates/carry/TSS/DM (USD)	0.58	0.13	1.89	-0.17
40	rates/momentum/time-series/DM (USD)	1.21	0.46	2.11	-0.27
41	rates/momentum/time-series/US (USD)	1.22	0.13	3.07	-0.18
42	rates/momentum/time-series/Europe (EUR)	0.91	0.28	2.64	-0.02
43	rates/volatility/carry/DM (USD)	1.57	0.15	3.38	-2.36
44	rates/volatility/carry/US (USD)	1.10	0.16	2.52	-4.61

Continued on next page

Continued from previous page

i	Name	SR	$\hat{\sigma}$ (SR)	$\mathcal{DD}^{\star}$	γ_1
45	rates/volatility/carry/Europe (EUR)	1.30	0.17	5.14	-0.50
46	rates/volatility/carry/Japan (JPY)	0.84	0.14	6.51	-2.59
47	currencies/carry/FRB/Global (USD)	0.89	0.06	4.45	-1.23
48	currencies/carry/FRB/DM (USD)	0.34	0.07	5.60	-0.89
49	currencies/carry/FRB/EM (USD)	0.96	0.28	5.24	-2.72
50	currencies/momentum/time-series/Global (USD)	0.64	0.13	4.22	-0.31
51	currencies/momentum/time-series/DM (USD)	0.44	0.08	3.51	0.04
52	currencies/momentum/time-series/EM (USD)	1.19	0.40	2.11	-0.65
53	currencies/value/economic model/DM (USD)	1.04	0.08	3.82	0.34
54	currencies/value/PPP/DM (USD)	0.66	0.02	4.90	-1.56
55	commodities/carry/FRB/Global (USD)	0.90	0.24	2.45	-0.12
56	commodities/carry/TSS/Global (USD)	2.65	0.27	1.97	-0.79
57	commodities/liquidity/Global (USD)	2.62	0.04	1.14	-0.33
58	commodities/momentum/cross-section/Global (USD)	0.39	0.06	4.19	-0.19
59	commodities/momentum/time-series/Global (USD)	0.52	0.30	4.63	-0.07

Remark 4 *In Tables 14 and 15 on page 106, we report a more exhaustive list of statistics including the skewness coefficient $\gamma_1^{\star}$ proposed by Lempérière* et al. *(2014a); the ratio $\mathcal{R}_{\sigma}$ between the volatility of negative returns and the volatility of positive returns; excess kurtosis γ_2; correlation ρ and the beta β with the long-only benchmark. In particular, we notice that some alternative risk premia present high kurtosis: equities/event/merger arbitrage, equities/reversal, equities/volatility, rates/volatilities, currencies/carry, currencies/value/PPP and commodities/liquidity.*

In Figures 23 and 24, we report the relationship between the skewness coefficient $\gamma_1^{\star}$ and the Sharpe ratio SR. For traditional risk premia, we verify the assumption of Lempérière *et al.* (2014a). We obtain a negative relationship between these two statistics, especially if we consider two categories: equities on the one hand and rates/credit on the other hand. However, the original empirical relationship[36] (12) is shifted[37]. If we consider alternative risk premia (Figure 24), we find a similar relationship as for rates and currencies (green dashed line). The relationship is inverted for commodities[38] whereas it is a U-shaped function for equities (solid blue line). In this last case, if we only consider ARP generic indices with negative skewness, we obtain a decreasing function, but with a lower slope[39] (dashed blue line).

Summary 14 *Empirical results show that alternative risk premia are very different in terms of risk measurement (volatility, skewness, kurtosis, drawdown). In particular, some of them present more risk than traditional risk premia (equities and bonds).*

3.2.3 Correlation analysis of ARP generic indices

ARP proponents believe that diversification is one of the largest benefits when investing in a portfolio of alternative risk premia for two reasons:

1. Including alternative risk premia expands the investment universe of traditional risk premia. For instance, if we consider the case of equities, we have one traditional

[36] Represented by the black solid line.

[37] This is higher for rates and credit (dashed green line) but lower for equities (dashed blue line). Again, the study period (January 2000 to December 2015) may explain this discrepancy.

[38] The behavior of commodity/liquidity and commodity/carry/TSS risk factors is particularly puzzling. One might wonder whether indices that composes these two generic backtests have been optimized.

[39] The slope is equal to 10% whereas the original value found by Lempérière *et al.* (2014a) is equal to 25%.

Figure 23: Relationship $\gamma_1^\star$/ SR for traditional risk premia

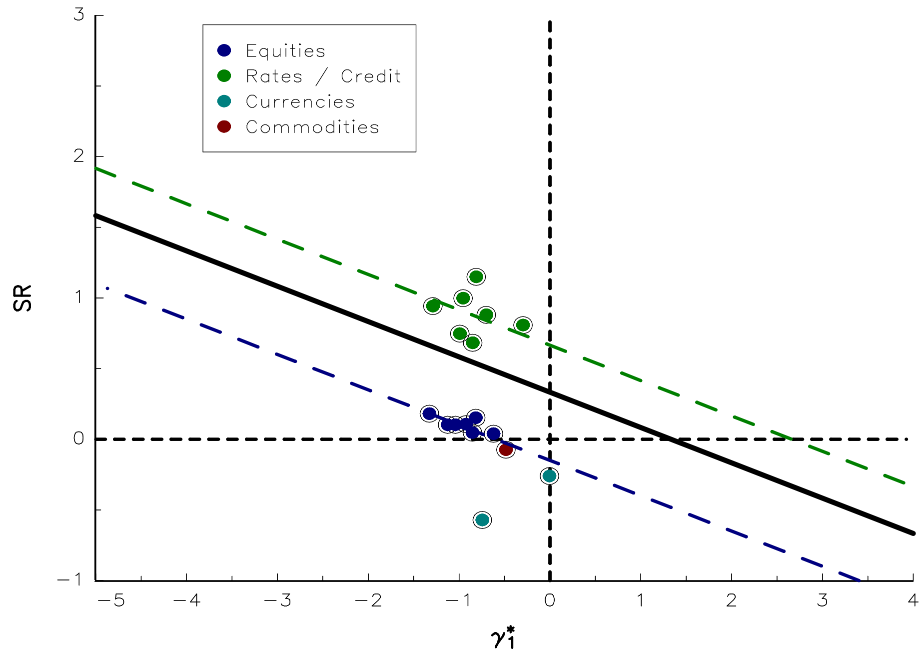
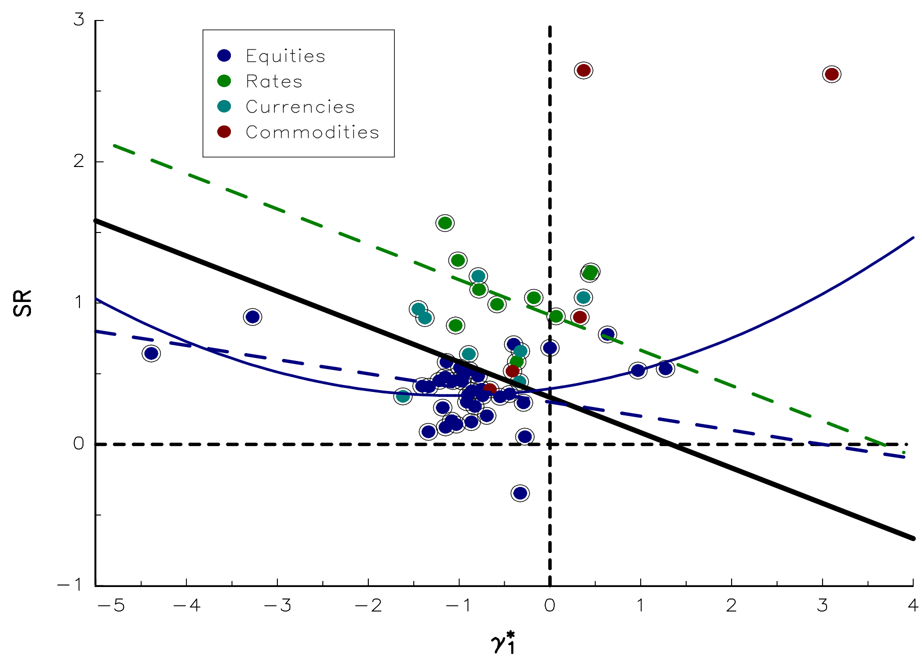


Figure 24: Relationship $\gamma_1^\star$/ SR for alternative risk premia

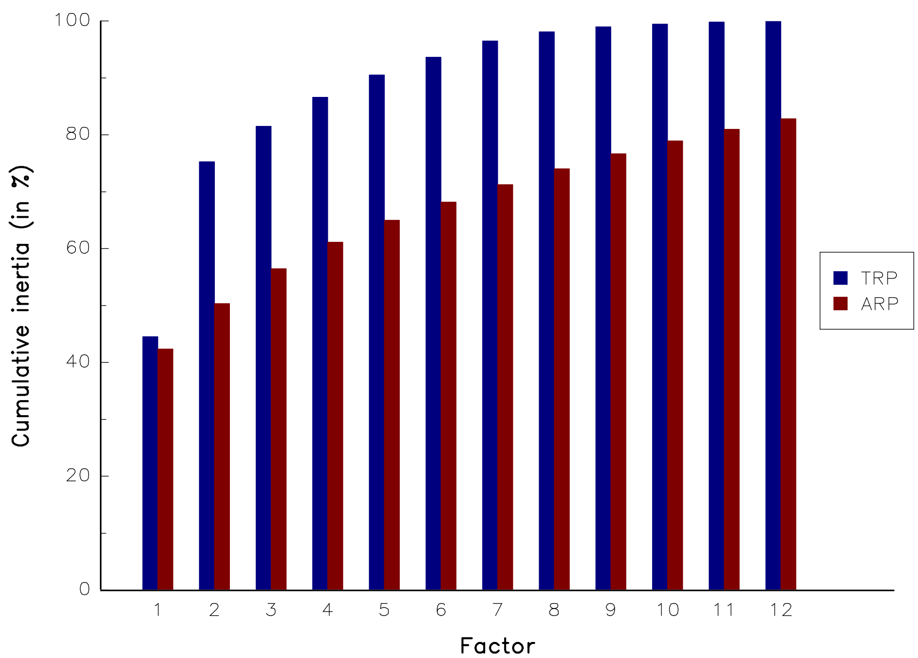


risk premium[40], but several alternative risk premia (carry, low volatility, momentum, quality, size, value, etc.).

2. The second reason is the decorrelation effect with traditional risk premia, in particular when alternative risk premia are in long/short format.

The first point is related to Figure 16, which shows that alternative risk premia complement traditional risk premia[41]. The second point is less clear cut. In the case of rates, currencies and commodities, the long/short format is the standard; this is clearly a result of the nature of the instruments in question (futures, forward contracts, swaps, options). This is not true for equities, because most products are in long-only format except when the instruments are futures on equity indices. To measure the diversification benefit, we conduct a principal component analysis on the two universes (TRP and ARP). For each universe, we report the cumulative inertia (in %) with respect to the number of PCA factors in Figure 25. We notice that the first PCA dimension explains about 40% of the total variance of the investment universe. With two dimensions, this figure becomes 75% for the TRP universe and 50% for the ARP universe. The analysis of the other dimensions confirms that the ARP universe provides more diversification than the TRP universe[42].

Figure 25: Principal component analysis



Remark 5 *This diversification is mitigated in the short term. Indeed, we notice that extreme negative returns tend to occur at the same time. For instance, we report the worst* 1% *of historical daily returns in Figures 26 and 27. We observe that most of these events took place in the same time periods. This is not surprising if we consider the drawdown premium assumption.*

Figure 26: Worst 1% of daily returns (traditional risk premia)

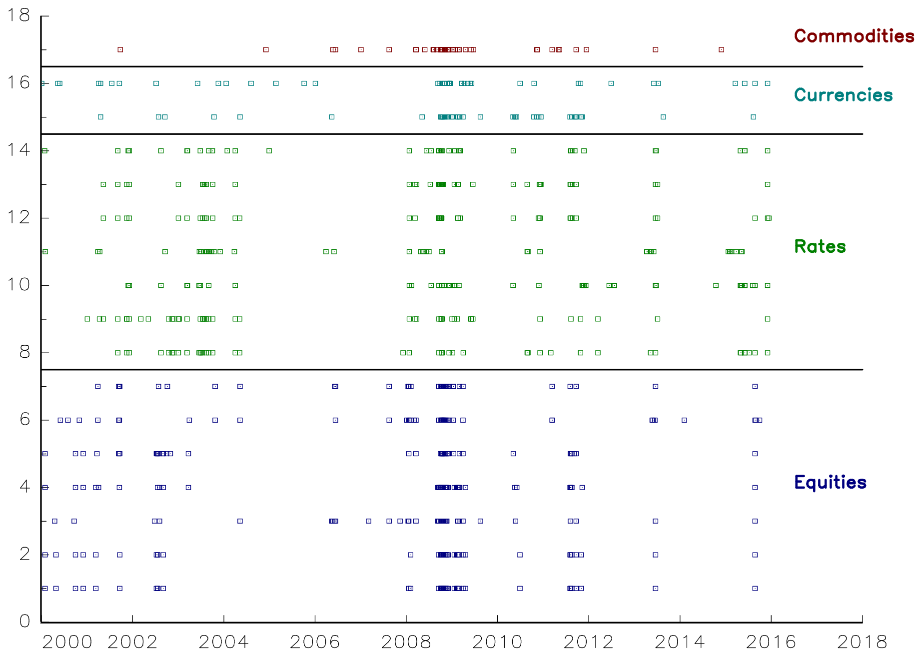
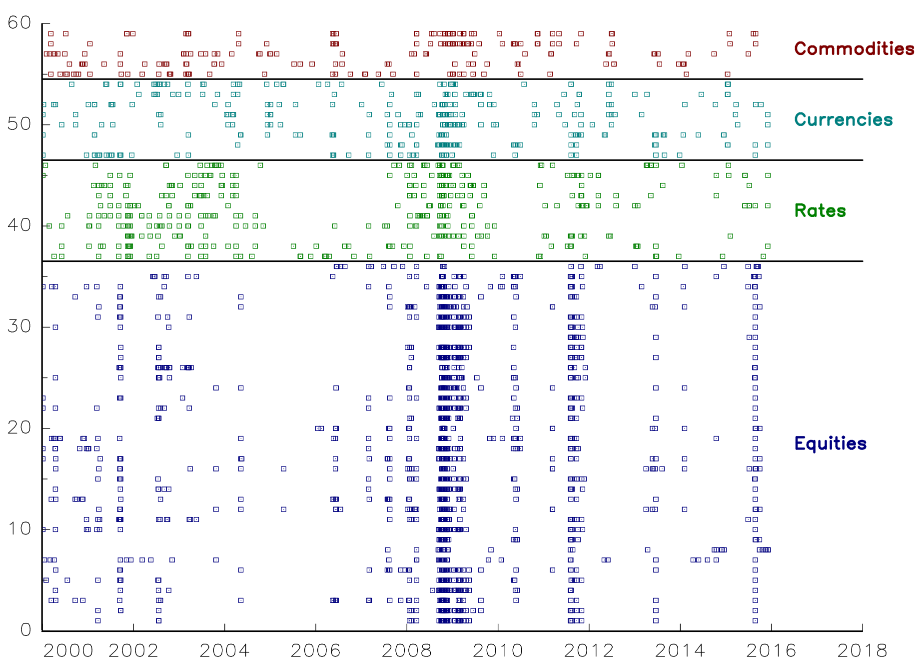


Figure 27: Worst 1% of daily returns (alternative risk premia)



Summary 15 *With alternative risk premia, we can benefit from volatility diversification, because the number of alternative risk factors is larger than the number of traditional risk premia and they present more significant independent dimensions.*

4 Diversified portfolios and alternative risk premia

4.1 Aggregation of skewness risk premia

Diversification is a financial concept that is extensively used in portfolio and risk management. The underlying idea is that the risk of the sum of portfolios X and Y is smaller than the sum of individual risks:

$$\mathcal{R}(X+Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$$

where $\mathcal{R}$ is the risk measure. For instance, if we consider the volatility, we have:

$$\begin{aligned} \sigma(X+Y) &= \sqrt{\sigma^2(X) + \sigma^2(Y) + 2\rho(X,Y)\,\sigma(X)\,\sigma(Y)} \\ &\leq \sigma(X) + \sigma(Y) \end{aligned}$$

If we wish to minimize the volatility, one solution consists in investing in assets or portfolios that are low or negatively correlated. This is why that such correlation is generally the statistical tool associated with the diversification concept in finance. However, correlation is a dependence measure, which is valid in a Gaussian world. This condition does not generally hold in finance, but we implicitly assume that the approximation makes sense for some assets (stocks, bonds, multi-assets). With alternative risk premia, we face a challenge because the component of high negative skewness is incompatible with the Gaussian assumption.

Let us consider an equally-weighted portfolio of the most skewed alternative risk premia. In Figures 28 and 29, we report the cumulative performance when we consider the five worst strategies in terms of skewness coefficient $\gamma_1^\star$ and drawdown ratio $\mathrm{DD}^\star$. In Figure 28, we consider both long-only and long/short risk premia, whereas we restrict the universe to long/short risk premia in Figure 29. The cumulative performance is expressed in total return in the first case and excess return in the second case. With long-only risk premia, we observe that EW-ARP portfolios have a volatility that is in line with the MSCI World index. However, they have a higher drawdown[43]. If we consider exclusively long/short risk premia, both volatility and drawdown are reduced. However, the drawdown ratio $\mathrm{DD}^\star$ is very high and larger than 8. Moreover, EW-ARP-LS portfolios have a skewness that is twice that of EW-ARP portfolios.

At first sight, these results are disturbing. In the case of the volatility risk measure, we find that diversification effects are higher with long/short risk premia than with long-only risk premia. This can easily be understood because correlations are lower for long/short strategies than for long-only strategies. In the case of skewness and drawdown risk measures, we obtain contrary results and diversification is lower with long/short risk premia. To understand this paradox, we must analyze the aggregation of skewness risks.

[40] This is the equity market portfolio.

[41] There are even serious doubts about the existence of some ARPs.

[42] For example, the 80% level is reached with three and 11 dimensions for TRP and ARP universes.

[43] The drawdown is equal to 57.46% for the MSCI World, 62.17% for the EW-ARP-$\gamma_1^\star$ portfolio and 67.52% for the EW-ARP-$\mathrm{DD}^\star$ portfolio.

Figure 28: Cumulative return of the ARP-EW portfolio

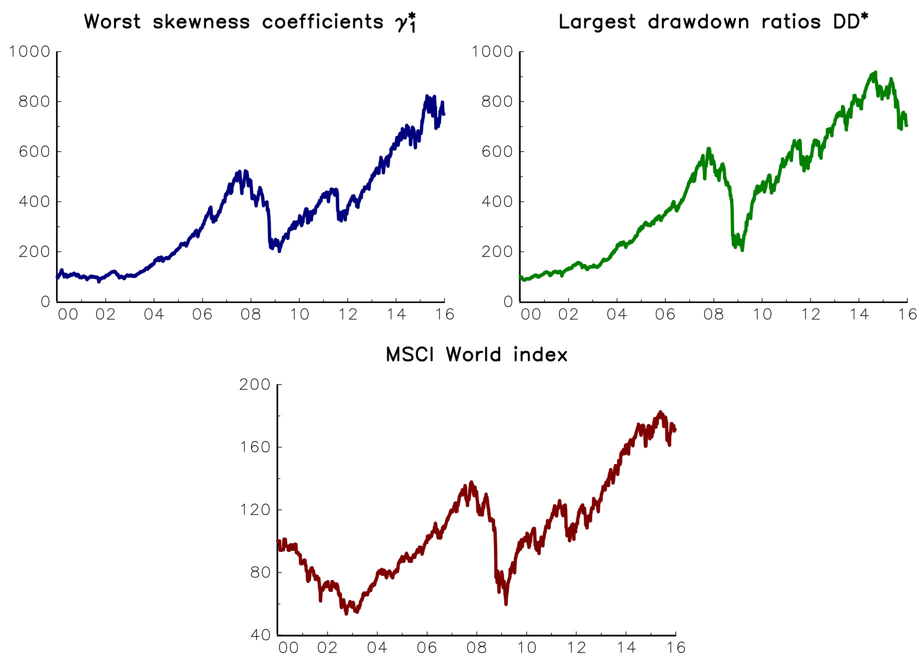
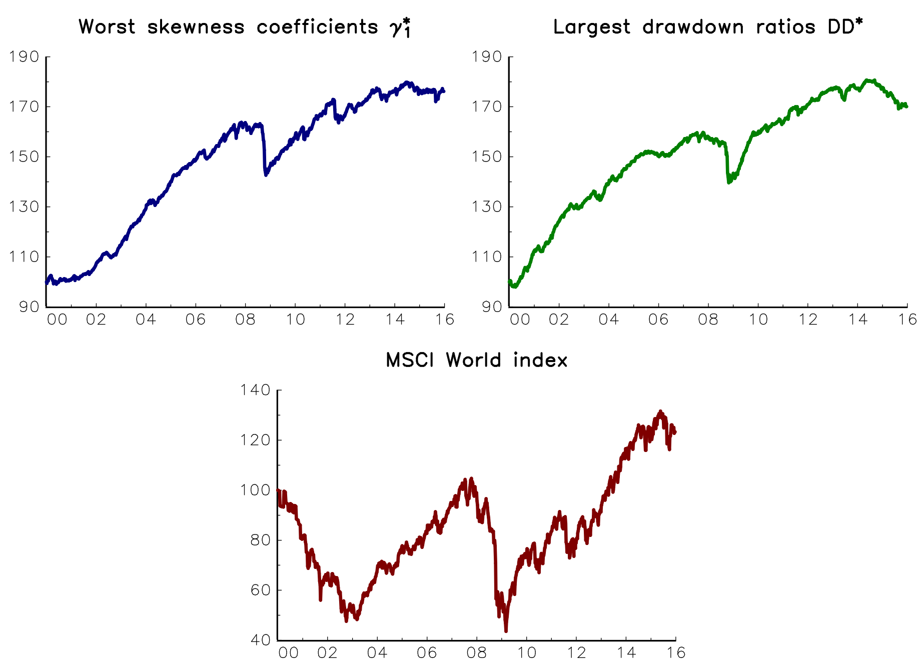


Figure 29: Cumulative return of the ARP-EW-LS portfolio



We consider two random variables X and Y, whose skewness is not equal to zero. In Appendix C.7, we demonstrate that the relationship between the aggregated skewness risk $\gamma_1(X+Y)$ is a complex function of individual skewness risks $\gamma_1(X)$ and $\gamma_1(Y)$. We remind readers that:

$$\gamma_1(X+Y) = \frac{\mu_3(X+Y)}{\sigma^3(X+Y)}$$

As shown previously, the relationship between $\rho(X,Y)$ and $\sigma(X+Y)$ is monotone and decreasing. In Section C.7.2 on page 103, we prove that $\mu_3(X+Y)$ is a decreasing function of the correlation $\rho(X,Y)$, but an increasing function of correlations $\rho(X^2,Y)$ and $\rho(X^2,Y)$. As a result, $\gamma_1(X+Y)$ is a decreasing function of the correlation $\rho(X,Y)$:

Effect	$\rho(X,Y)$	$\rho(X^2,Y)$	$\rho(X,Y^2)$
$\sigma(X+Y)$	$+$	0	0
$\mu_3(X+Y)$	$-$	$+$	$+$
$\gamma_1(X+Y)$	$-$	$+$	$+$

However, the previous analysis does not account for the relationships between $\rho(X,Y)$, $\rho(X^2,Y)$ and $\rho(X^2,Y)$. Indeed, the impact of $\rho(X^2,Y)$ and $\rho(X^2,Y)$ on $\mu_3(X+Y)$ is generally more important than the impact of $\rho(X,Y)$. As a result, the consequence of the two last correlations are highly significant.

Let us consider an example to illustrate why skewness aggregation is a complex process. We assume that the opposite of the random vector (X,Y) follows a bivariate log-normal distribution[44]. This probability distribution was chosen because it exhibits non-zero skewness and we have an analytical formula[45] for $\gamma_1(X+Y)$. By using the parameters $\mu_X = \mu_Y = \sigma_X = \sigma_Y = 0.5$, we obtain the top/left panel in Figure 30, which shows the evolution of $\gamma_1(X+Y)$ with respect to the correlation parameter ρ of the bivariate log-normal distribution. We notice that it is an increasing function until ρ is equal to -30%, and then a decreasing function. To understand this behavior, we report the contributions of second and third moments in the bottom/left panel:

$$C_3(\rho) = \frac{\mu_3(X+Y;\rho)}{\mu_3(X+Y;1)}$$

and:

$$C_2(\rho) = \frac{\sigma^3(X+Y;\rho)}{\sigma^3(X+Y;1)}$$

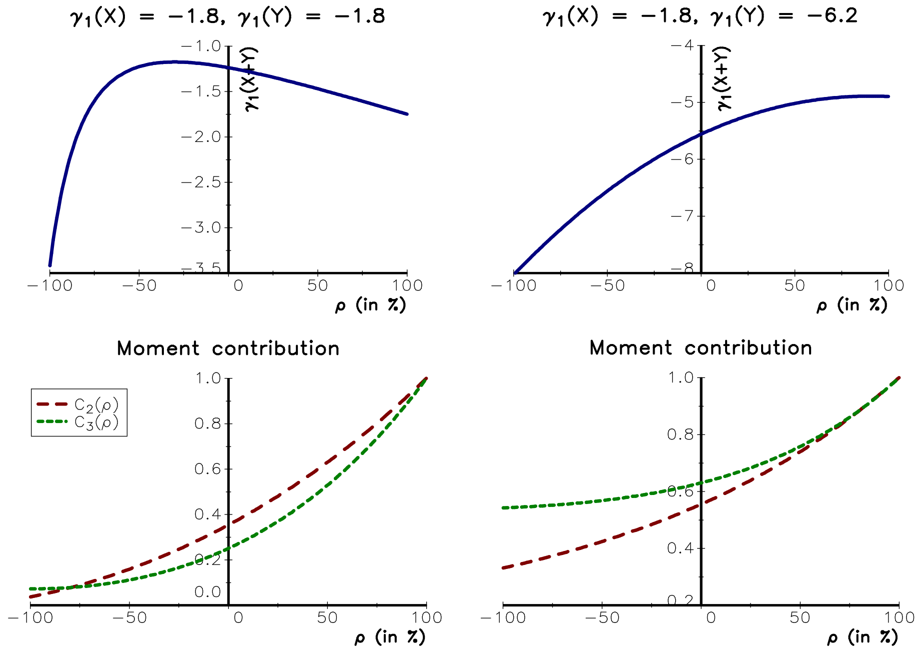
These contributions represent the ratio between the moment with parameter ρ and the moment when the dependence function is equal to the upper Fréchet copula[46] $\mathbf{C}^+$. It follows that:

$$\gamma_1(X+Y;\rho) = \frac{C_3(\rho)}{C_2(\rho)} \times \gamma_1(X+Y;1)$$

We notice that there is a limit of the diversification for the third moment, which is not the case for the second moment. Indeed, when $\rho < -70\%$, the ratio $C_3(\rho)$ reaches a floor whereas the volatility continues to decrease. This behavior is even more apparent when X and Y have different skewness coefficients. For instance, the top/right panel corresponds to the case $\sigma_Y = 1$, whereas the value of the other parameters remains the same. In the bottom/right panel, the ratio $C_3(\rho)$ is nearly a horizontal line when $\rho < -50\%$.

[44] We take the opposite in order to obtain negative skewness coefficients.
[45] It is given in Appendix C.7.3 on page 103.
[46] This case corresponds to $\rho = 1$.

Figure 30: Aggregation of skewness



These examples show that there is a maximum diversification if we consider the skewness risk measure. The problem is twofold. First, volatility diversification is a limiting factor for skewness diversification. Indeed, by decreasing the volatility, we implicitly increase the skewness coefficient, all other things being equal. Second, the diversification of the third moment is an issue too, in that it is extremely difficult to hedge large losses. How can we explain this discrepancy between the behavior of the second moment and the behavior of the third moment? The answer lies in understanding the stochastic dependence between skewness risk premia. Analyzing skewness aggregation using probability distributions is equivalent to considering a reduced-form model. However, this approach ignores the fact that asset prices including alternative risk premia are stochastic processes. When a stochastic process exhibits high skewness, we generally break it down into a trend component, a Brownian component and a singular component. Unlike regular and irregular variations that are easy to diversify, it is difficult to hedge discontinuous variations. In their simplest form, these singular variations are jumps. The worst-case scenario concerning skewness aggregation is thus to build a well-diversified portfolio by dramatically reducing volatility while the probability of jumps remains high. This explains the results obtained in Figures 28 and 29. Indeed, ARP-EW-LS portfolios have greater skewness risk than ARP-EW portfolios, because volatility diversification is higher when we consider long/short strategies instead of long-only strategies.

Summary 16 *The aggregation of skewness risk premia is an issue in terms of risk management. Indeed, the accumulation of skewed strategies would expose the investor to a risk of large drawdowns, which is difficult to mitigate by volatility diversification. More precisely, volatility diversification and skewness diversification are not always compatible. In particular, linear correlation is not the right statistical tool to perform the aggregation of alternative risk premia, which have very high skewness coefficients.*

4.2 Payoff of alternative risk premia

We are interested in the behavior of alternative risk premia with respect to traditional risk premia. To examine this point in greater depth, we estimate the payoff function of each ARP with respect to a given benchmark. Let $R_t(x)$ and $R_t(b)$ be the returns of the ARP x and the benchmark b. If the dependence function between $R_t(x)$ and $R_t(b)$ is the lower Fréchet copula $\mathbf{C}^-$, we obtain:

$$R_t(x) = f(R_t(b))$$

where f is a decreasing function. We obtain a similar result in the case of the upper Fréchet copula $\mathbf{C}^+$, but the function f is now increasing. We denote by $R_{t:T}(x)$ and $R_{t:T}(b)$ the order statistics associated with a sample of $R_t(x)$ and $R_t(b)$ with length T. The relationship above then becomes:

$$\sum_{i=1}^{t} R_{i:T}(x) = g\left(\sum_{i=1}^{t} R_{i:T}(b)\right)$$

where g is a concave function if the copula is $\mathbf{C}^-$ (and, conversely, g is a convex function if the copula is $\mathbf{C}^+$). We introduce the conditional order statistic $R_{t:T}(x \mid b)$ of the ARP x, where the ordering $t:T$ is calculated with respect to the returns of benchmark b. We can analyze the dependence function between $R_t(x)$ and $R_t(b)$ by considering the shape of the function $h(u)$ defined as follows:

$$h\left(\frac{t}{T}\right) = \sum_{i=1}^{t} R_{t:T}(x \mid b)$$

This approach has the advantage of normalizing the marginals. In Figure 31, we represent the function $h(u)$ when the alternative risk premium is the equities/volatility/carry/US strategy and the benchmark is the MSCI ACWI index. The solid blue line corresponds to the cumulative returns $\sum_{i=1}^{t} R_i(x)$, the dashed lines represent the function $g(u)$ when the copula function is $\mathbf{C}^-$ and $\mathbf{C}^+$, and the red line shows the conditional dependence function $h(u)$.

We estimate the payoff function between $R_{t:T}(b)$ and $R_{t:T}(x \mid b)$ by considering a non-parametric approach. To this end, we apply a non-parametric regression with a spline kernel on the monthly returns. We determine the 10% and 90% confidence intervals by simulating bootstrapping samples and fitting a non-parametric quantile regression. In the case of the previous example, results are shown in Figure 32. The solid black line represents the estimated payoff function, whereas the dashed lines corresponds to the confidence interval. In the case of the equities/volatility/carry/US strategy, we recognize a short-put option profile.

We have applied the previous analysis to the traditional and alternative risk premia by considering two benchmarks: the MSCI ACWI index (equities) and the Barclays Global Agg Govt index (rates). Results are given on pages 115–121. These figures give the payoff function estimated with the non-parametric regression and the spline kernel. We can then classify the various alternative risk premia with respect to standard option profiles. However, this classification is not an easy task. In particular, the option profile may change from one region to another and we may note considerable uncertainty when analyzing confidence intervals. Nevertheless, this classification remains very informative for investors, whose aim is to build a diversified portfolio.

Figure 31: Conditional dependence $h(u)$ for the equities/volatility/carry/US strategy

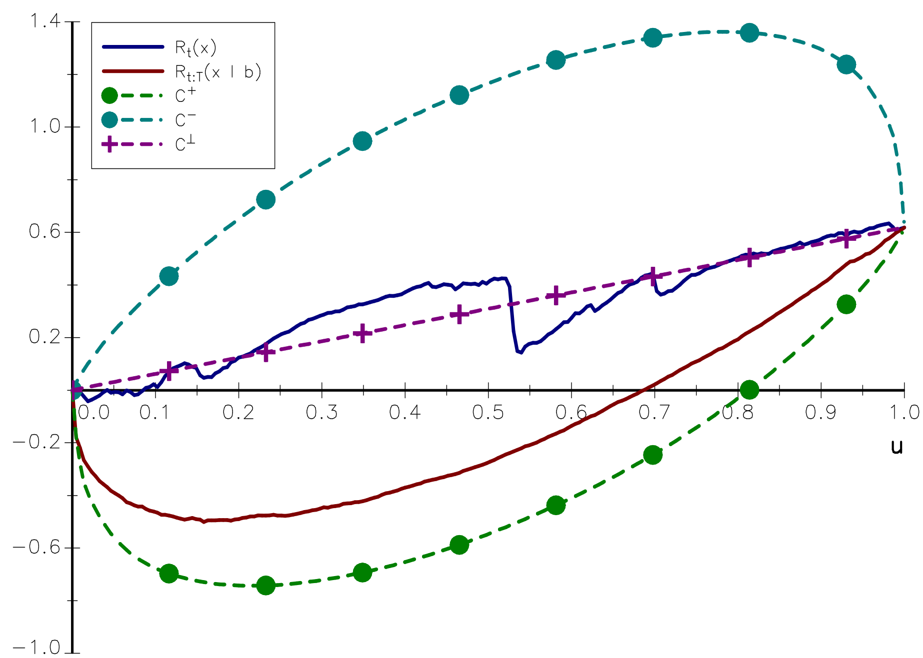


Figure 32: Payoff function estimation for the equities/volatility/carry/US strategy

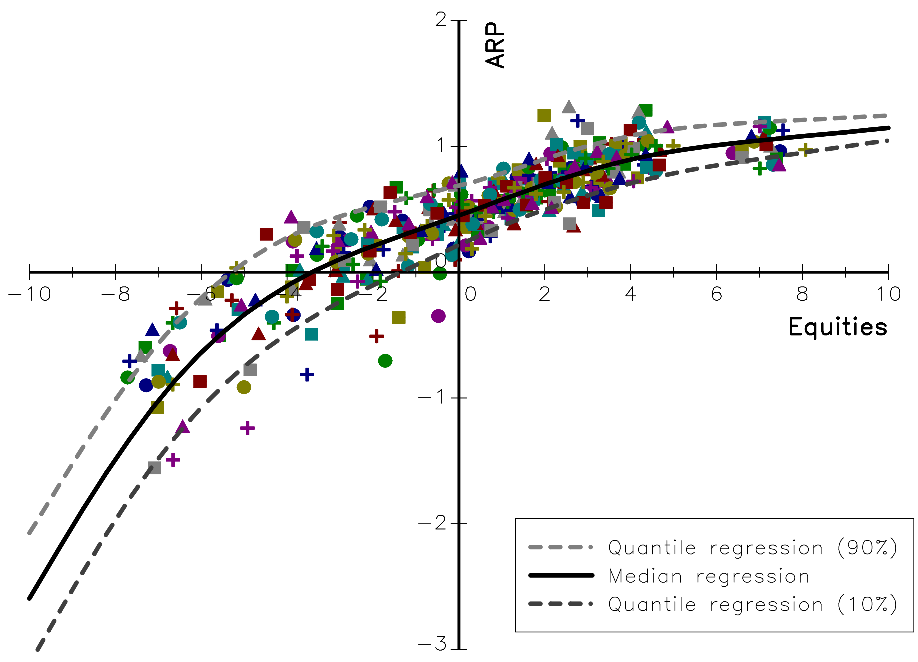


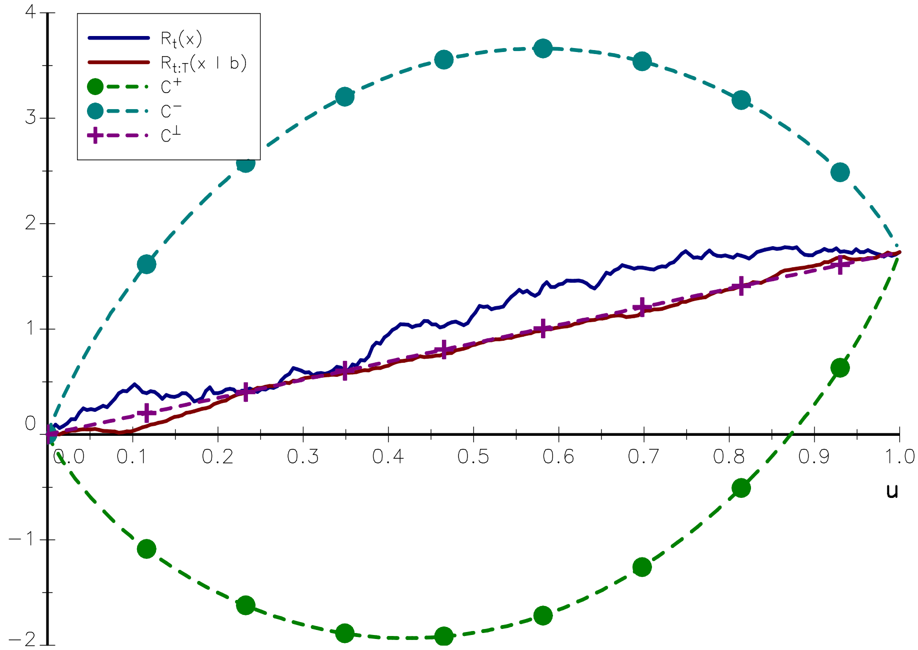
Table 6: Payoff functions of alternative risk premia

Asset class	ARP	Payoff function	
		Equities	Rates
Equities	Carry	long-only	short-call
	Event (long)	short-put	short-call
	Event (long/short)	$\perp$	$\perp$
	Growth	long-only	short-call
	Low volatility	long-only	short-call
	Momentum	long-only	short-call
	Quality	long-only	short-call
	Reversal	short-put*	long-call*
	Value	leveraged	short-call
	Volatility (carry)	short-put	short-call*
	Volatility (term structure)	long-put	long-call
Rates	Carry	long-put	long-only
	Momentum	long-straddle*	long-only
	Volatility	long-call	short-straddle
Currencies	Carry	long-only	short-call
	Momentum	long-strangle	$\perp$
	Value	long-strangle*	$\perp$
Commodities	Carry	$\perp$	short-put*
	Liquidity	$\perp$	short-put*
	Momentum (cross-section)	short-straddle*	long-only*
	Momentum (time-series)	short-risk-reversal*	long-put*

A summary of the results is given in Table 6. Let us first consider the case of the equities benchmark. If we consider long-only equities strategies, the payoff profile is long-only with more or less leverage. The case of the equities/event strategy is interesting because the payoff differs if we consider a long-only or a long/short format. In a long-only format, it resembles a short-put option, whereas it is more of an independent payoff in a long/short format. This difference was identified by Cazalet and Roncalli (2014), who observed that the long/short equities/momentum/cross-section strategy exhibits a short-call option profile, whereas it is a linear payoff in a long-only format. For some payoff functions, we add an asterisk (*) to indicate that the confidence interval exhibits significant uncertainty around the median estimation. For instance, this is the case for the equities/reversal strategy. Other payoff functions are well identified, such as the equities/volatility/carry strategy, which is a short-put option. Rates/currencies strategies present more diversified option functions. For example, the rates/carry, rates/volatility and currencies/momentum have long-put, long-call and long-straddle profiles, respectively. The case of commodities strategies is more complex, because they exhibit exotic profiles, but these option patterns are generally not important when we analyze confidence intervals. This is confirmed when we plot the conditional dependence $h(u)$, which resembles the dependence function of the product copula $\mathbf{C}^{\perp}$ (see, for instance, Figure 33 for the commodities/carry/FRB strategy).

If we consider the rates benchmark, most equities strategies have a short-call option profile. The exceptions are equities/reversal (long-call) and equities/volatility/term structure (long-call). More surprisingly, rates/carry and rates/momentum have a strong positive dependence on the rates benchmark. Currencies strategies are more or less independent,

Figure 33: Conditional dependence $h(u)$ for the commodities/carry/FRB strategy



with the exception of the carry strategy that exhibits a short-call profile. Finally, the option patterns of commodities strategies are not significant.

Summary 17 *Alternative risk premia may exhibit non-linear payoffs with respect to a long position on equities or rates. It appears that only a few of them may be viewed as pure risk premia*[a]*. Some strategies are also independent from equities and rates, such as the commodities alternative risk premia.*

[a]For instance, this is the case for equities/volatility/carry, and, to a lesser extent, equities/event/long and equities/reversal.

4.3 Portfolio construction with alternative risk premia

Given that alternative risk premia may exhibit high skewness and non-linear payoffs, traditional methods used to build a diversified portfolio (Markowitz, risk parity, volatility target, etc.) are not adapted[47]. The main reason for this is that volatility and correlation risk measures are not suitable to measure risk and diversification in this context[48]. For instance, no one would use mean-variance optimization within a universe of call and put options. The academic answer to this challenge is to consider a utility maximization problem, which accounts for high-order moments (Jurczenko and Maillet, 2006; Martellini and Ziemann, 2010). However, this approach is difficult to implement in practice because of instability issues concerning high-order moment estimation.

Portfolio construction with a universe of alternative risk premia can only be pragmatic and heuristic. Firstly, it is important to classify ARP strategies into homogenous clusters

[47]This issue is not specific to alternative risk premia. It is generally the case for all portfolios of non-linear instruments, e.g. portfolios of hedge funds.

[48]See for instance Leland (1999), Goetzmann *et al.* (2002), Broadie *et al.* (2009) and Malamud (2014).

(skewness risk premia versus market anomalies, short-put versus straddle option profiles, low volatility versus high volatility, etc.). Second, it makes sense to balance exposure to the various clusters in order to control the portfolio's drawdown. Finally, it is essential to conduct scenario analysis in order to understand the future behavior of the ARP portfolio.

Summary 18 *Mean-variance portfolio optimization and traditional allocation models are not relevant when building a diversified portfolio of alternative risk premia, because they do not address the issue of skewness aggregation and non-linear payoffs.*

5 Understanding the performance of hedge fund strategies

The emergence of the alternative risk premia paradigm should enable us to better understand the performance of hedge fund strategies. Since the seminal work of Fung and Hiseh (1997), academics have analyzed the behavior of the hedge fund industry by considering time-varying exposure to particular risk factors. In particular, the framework developed by Hasanhodzic and Lo (2007) consists in breaking hedge fund performance down into several components: traditional beta, alternative beta[49] and alpha. The traditional beta is the part of the performance explained by the constant exposure to risk factors, whereas the alternative beta is the part of the performance explained by the dynamic exposure to the same risk factors. The alpha is the remaining part of the hedge fund's performance, which is not explained by risk factors. Until now, many academic studies have considered a set of risk factors, principally comprising traditional risk premia (Hasanhodzic and Lo, 2007; Roncalli and Teiletche, 2008; Roncalli and Weisang, 2009; Amenc *et al.*, 2010). In this section, we extend the set of risk factors by including alternative risk premia. The objective is to determine whether alternative risk factors may help us to understand the risk/return profile of hedge fund strategies.

5.1 The statistical framework

Traditionally, hedge fund returns are broken down by considering the following linear model:

$$R_{i,t} = R_{f,t} + \sum_{j=1}^{n_{\mathcal{F}}} \beta_{i,t}^{j} \mathcal{F}_{j,t} + \varepsilon_{i,t} \tag{15}$$

where $R_{i,t}$ is the monthly return of the hedge fund strategy i, $R_{f,t}$ is the risk-free rate, $\mathcal{F}_{j,t}$ is the excess return of factor j, $\beta_{i,t}^{j}$ is the exposure of hedge fund strategy i to factor j and $\varepsilon_{i,t}$ is a white noise process. The issue is then to estimate the time-varying exposures $\beta_{i,t}^{j}$. Two approaches are generally adopted:

1. In the first approach, the dynamic allocation is estimated using rolling OLS. This means that the calibration of parameters $\beta_{i,t}^{j}$ is performed using the period $[t-h, t]$, where h is the rolling window.

2. The second approach assumes that the exposure to one risk factor varies from month to month according to a random walk:

$$\beta_{i,t}^{j} = \beta_{i,t-1}^{j} + \eta_{i,t}^{j}$$

[49] We use here the academic term "*alternative beta*" to describe time-varying positions as implemented by hedge funds. Some professionals also use this term as a synonym for alternative risk premia.

where $\eta_{i,t}^{j}$ is a white noise process. In this case, parameters $\beta_{i,t}^{j}$ are estimated using the Kalman filter.

One of the issues with these two approaches is the over-fitting bias when the number of risk factors is large. This is true in our case, because we will consider the universe of traditional and alternative risk premia as the set of risk factors. This is why we prefer to adopt a third approach to estimate the exposure. When dealing with a large universe of risk factors, the main issue is the selection procedure of risk factors. For this, we follow the approach of Giamouridis and Paterlini (2010), who proposed using the lasso method to estimate time-varying exposure.

Remark 6 *We can also supplement the lasso approach by a qualitative selection of relevant risk premia done by hedge fund analysts in order to reduce the number of risk premia and mitigate the risk of spurious results.*

5.1.1 The lasso method

Let us consider the linear regression model:

$$Y = X\beta + U$$

where Y is the $(T \times 1)$ vector of endogenous data, X is a $(T \times n_{\mathcal{F}})$ matrix of exogenous data, β is a $(n_{\mathcal{F}} \times 1)$ vector of parameters and U is the vector of residuals. When the number of explanatory variables is large, we may use the penalized ridge regression:

$$\hat{\beta}(\lambda) = \arg\min (Y - X\beta)^\top (Y - X\beta) + \lambda \beta^\top \beta$$

We notice that the ridge regression may be formulated as the following problem:

$$\begin{aligned} \hat{\beta}(\lambda) &= \arg\min (Y - X\beta)^\top (Y - X\beta) \\ & \text{u.c. } \sum_{j=1}^{n_{\mathcal{F}}} \beta_j^2 \leq \tau \end{aligned}$$

Tibshirani (1996) proposes a variant of the ridge regression called the lasso method when the penalty function is replaced by the L_1 norm:

$$\begin{aligned} \hat{\beta}(\tau) &= \arg\min (Y - X\beta)^\top (Y - X\beta) \\ & \text{u.c. } \sum_{j=1}^{n_{\mathcal{F}}} |\beta_j| \leq \tau \end{aligned}$$

The lasso approach has the advantage of producing more sparsity than the ridge approach.

5.1.2 Application with regard to the analysis of hedge fund returns

In this section, we consider the well-known hedge fund indices from two providers: HFR and EDHEC. For each index provider, we consider the following categories:

1. HFR
   Fund Weighted Composite index (HFRI), Macro:Systematic Diversified index (CTA), Event Driven: Distressed/Restructuring index (DS), Event Driven (Total) index (ED), Equity Hedge (Total) index (EH), Emerging Markets (Total) index (EM), Equity Hedge: Equity Market Neutral index (EMN), Event Driven: Merger Arbitrage index (MA), Macro (Total) index (MAC), Relative Value (Total) index (RV), Equity Hedge: Short Bias index (SB), Fund of Funds Composite index (FOF);

2. EDHEC indices
   Convertible Arbitrage index (CA), CTA Global index (CTA), Distressed Securities index (DS), Event Driven index (ED), Emerging Markets index (EM), Equity Market Neutral index (EMN), Fixed Income Arbitrage index (FIA), Global Macro index (GM), Long/short Equity index (LSE), Merger Arbitrage index (MA), Relative Value index (RV), Short Selling index (SB), Funds of Funds index (FOF).

For traditional risk premia, we use the risk factors that are commonly employed for hedge fund replication: equity exposure to the S&P 500 index (SPX), a long/short position between the Russell 2000 index and the S&P 500 index (RTY), a long/short position between the Eurostoxx 50 index and the S&P 500 index (SX5E), a long/short position between the TOPIX index and the S&P 500 index (TPX), a long/short position between the MSCI EM index and the S&P 500 index (MXEF), exposure to the 10Y US Treasury bond (UST), two FX positions between the euro and the US dollar (EUR) and between the yen and the US Dollar (JPY), exposure to high yield bonds (HY), exposure to emerging bonds (EMBI), exposure to commodities (GSCI) and exposure to gold (GOLD)[50]. All these risk factors are measured in excess return with respect to the Libor USD 1M, except for the four examples of long/short equity exposure.

For alternative risk premia, we consider the 59 risk factors defined in Section two. All these risk factors are defined in terms of excess return, except some long-only equity risk factors. This is why we transform the latter group in a long/short format by adding a short position on the appropriate benchmark. For each region indicated in brackets, we use the following benchmarks: the MSCI ACWI index (Global), the MSCI World index (DM), the MSCI Emerging Markets index (EM), the MSCI USA index (US), the MSCI Europe index (Europe), the MSCI Japan index (Japan) and the MSCI Asia Pacific index (Asia Pacific).

We notice that the set of risk factors consists of 12 traditional and 59 alternative risk premia. Analyzing hedge fund returns with respect to 71 risk factors is not realistic and may produce over-fitting results. To solve this issue, we apply the lasso method as a variable selection procedure. From a practical point of view, we rewrite the model (15) in the following way:

$$\tilde{R}_{i,t} = R_{f,t} + \sum_{j=1}^{n_{\mathcal{F}}} \tilde{\beta}_i^j \tilde{\mathcal{F}}_{j,t} + u_{i,t} \tag{16}$$

where $\tilde{R}_{i,t}$ and $\tilde{\mathcal{F}}_{j,t}$ are the standardized values of $R_{i,t}$ and $\mathcal{F}_{j,t}$ divided by their standard deviation. In this case, the relationship between β_i^j and $\tilde{\beta}_i^j$ is:

$$\beta_i^j = \frac{\sigma\left(\mathcal{F}_{j,t}\right)}{\sigma\left(R_{i,t}\right)} \tilde{\beta}_i^j$$

The lasso estimator of $\left(\tilde{\beta}_i^1, \ldots, \tilde{\beta}_i^{n_{\mathcal{F}}}\right)$ is defined by the following optimization problem[51]:

$$\begin{aligned} \tilde{\beta}_i\left(\tau\right) &= \arg\min \sum_{t=1}^{T} \left(\tilde{R}_{i,t} - R_{f,t} - \sum_{j=1}^{n_{\mathcal{F}}} \tilde{\beta}_i^j \tilde{\mathcal{F}}_{j,t}\right)^2 + \lambda \sum_{j=1}^{n_{\mathcal{F}}} \left(\tilde{\beta}_i^j\right)^2 \\ & \text{u.c.} \quad \sum_{j=1}^{n_{\mathcal{F}}} \left|\tilde{\beta}_i^j\right| \leq \tau \end{aligned}$$

[50]These last four instances of exposure correspond to the BOFA ML US HY MASTER II index, the JPM EMBI index, the S&P GSCI Commodity TR index and the S&P GSCI Gold TR index.

[51]We introduce a low ridge regularization where λ is equal to 10^{-6} in order to be sure to obtain a positive-definite problem.

To obtain the estimated exposure, we apply the transformation:

$$\hat{\beta}_i^j(\tau) = \frac{\sigma(\mathcal{F}_{j,t})}{\sigma(R_{i,t})}\tilde{\beta}_i^j(\tau)$$

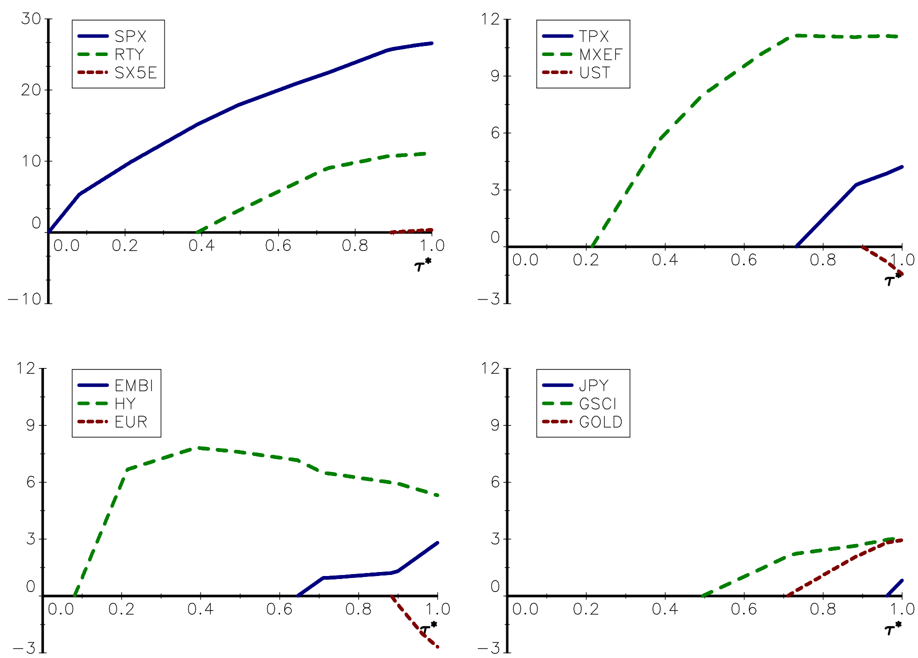
We then introduce the statistic $\tau^\star$ defined by:

$$\tau^\star = \frac{\sum_{j=1}^{n_{\mathcal{F}}}\left|\hat{\beta}_i^j(\tau)\right|}{\sum_{j=1}^{n_{\mathcal{F}}}\left|\hat{\beta}_i^j(\infty)\right|}$$

We may interpret this statistic as the shrinkage measure of the lasso model with respect to the OLS model[52].

We estimate Model (16) for the HFRI index with traditional risk premia over the entire study period (January 2000 to December 2015). Results are reported in Figure 34. The lasso method selects all the risk factors when $\tau^\star$ is equal to one. Then, it successively deletes JPY, UST, SX5E, etc. We finally obtain the following ordering of risk factors: (1) SPX, (2) HY, (3) MXEF, (4) RTY, (5) GSCI, (6) EMBI, (7) GOLD, (8) TPX, (9) EUR, (10) SX5E, (11) UST and (12) JPY. This means that the most important risk factor for understanding the performance of the hedge fund industry during the last 15 years is a long exposure on equity in developed markets. The other three main risk factors are high yield (or credit), equity in emerging markets and equity small caps exposure.

Figure 34: Lasso coefficients $\hat{\beta}^j$ (in %) for HFRI monthly returns



If we include the alternative risk premia in the set of explanatory variables, the ordering of the most important risk factors becomes: (1) SPX, (2) HY, (3) equities/growth/US,

[52]This is equal to one when τ is greater than $\tau_{\mathrm{OLS}} = \sum_{j=1}^{n_{\mathcal{F}}}\left|\hat{\beta}_i^j(\infty)\right|$ whereas it is equal to zero when τ is equal to zero. Moreover, it is a decreasing function of τ.

(4) equities/low volatility/EM, (5) MXEF, (6) equities/volatility/carry/US, (7) currencies/carry/FRB/EM, (8) equities/event/merger arbitrage/DM, (9) equities/low volatility/Japan, (10) GSCI, (11) equities/ momentum/cross-section/Europe, (12) commodities/momentum/time-series, etc. The two most important risk factors remain exposure to equity in developed markets and credit. However, we notice that the next 10 risk factors are mainly alternative risk premia. For instance, it seems that hedge fund managers have favored low volatility stocks in emerging markets and have invested in currency carry trade and short volatility strategies. Momentum strategies are also used in equities and commodities. More surprising is the low representation of alternative risk premia based on rates. Indeed, the first fixed-income strategy ranks 42^{nd} and is the carry/FRB risk factor.

The previous analysis is valid for the entire period 2000-2015. However, we notice that there is a hiatus in 2008 and that the results are not the same before and after the financial crisis. For instance, exposure to small caps stocks (RTY) is more pronounced before 2008, whereas exposure to credit (HY) increased significantly after 2008. With regards to alternative risk factors, the low volatility factor concerns principally EM stocks before 2008, whereas it focuses on DM stocks after 2008. The equity/volatility/carry factor does not exclusively concern the US market, but also applies to the European market after the financial crisis. Curiously, we do not notice a change in exposure to the currencies/carry risk factor.

Summary 19 *The lasso approach provides a robust method for analyzing hedge fund returns when the number of risk factors is high. It enables us to combine traditional and alternative risk premia and to illustrate how hedge funds are exposed to asset classes.*

5.2 Static in-sample analysis

In this paragraph, we perform a systematic analysis of monthly hedge fund returns by applying the lasso approach to the entire study period (January 2000 to December 2015). In Tables 7 and 8, we report the $\mathbf{R}^2$ statistic for four distinct cases:

1. We consider the 12 traditional risk premia (TRP).
2. The risk factors are the 59 alternative risk premia (ARP).
3. The long-only equity exposure (SPX) is added to the set of the 59 alternative risk premia (SPX + ARP).
4. In the last case, the set of risk factors consists of both traditional and alternative risk premia (TRP + ARP).

The results suggest that a significant part of the variability of some hedge fund strategies may be explained by traditional and alternative risk premia. This is the case for global, event-driven, equity hedge (or long/short equity), emerging markets, relative value and short bias indices, whose $\mathbf{R}^2$ with 10 risk factors is greater than 75%. The CTA strategy is interesting because the $\mathbf{R}^2$ statistic increases sharply when we include alternative risk premia. However, the results are disappointing, because the $\mathbf{R}^2$ statistic remains low. This may be because of the static approach, which produces constant exposure. In Figures 35 and 36, we report the split between the number of selected traditional risk factors and the number of selected alternative risk factors when we consider the 10-factor model. It follows that alternative risk factors surpass traditional risk factors in terms of frequency.

Remark 7 *Some of our results are in line with those obtained by Maeso and Martellini (2016). In particular, they found that "more dynamic and/or less directional strategies such as CTA Global, Equity Market Neutral, Fixed Income Arbitrage and Merger Arbitrage strategies are harder to replicate than more static and/or more directional strategies such as long-short equity or short selling".*

Summary 20 *The in-sample analysis of hedge fund returns shows that two traditional risk premia (DM equities, credit) are the main risk factors for the last* 15 *years. The other significant risk factors are essentially alternative risk premia. Moreover, they help to improve the return analysis of the CTA strategy.*

Table 7: In-sample $\mathbf{R}^2$ (in %) for HFR indices (2000-2015)

Strategy	TRP		ARP		SPX + ARP		TRP + ARP	
	5F	10F	5F	10F	5F	10F	5F	10F
HFRI	81.0	85.4	47.0	78.0	73.8	86.2	81.0	86.9
CTA	15.9	24.8	21.2	42.7	37.0	46.1	37.0	46.1
DS	60.1	62.8	41.3	50.1	46.3	59.7	60.1	67.4
ED	78.1	80.7	32.8	72.5	59.1	78.5	78.1	81.7
EH	85.3	87.0	57.7	80.8	81.3	85.8	85.3	88.9
EM	88.9	89.4	57.9	76.0	70.2	81.7	88.9	89.9
EMN	20.9	22.8	31.1	52.2	31.1	52.2	31.1	52.2
MA	45.7	50.2	19.2	54.2	49.1	60.5	49.1	63.0
MAC	30.0	35.0	25.2	49.3	28.7	58.3	35.8	58.3
RV	66.5	73.3	61.1	69.9	61.2	69.9	66.5	74.9
SB	67.0	70.0	68.3	74.4	81.8	85.3	81.8	85.3
FOF	63.3	68.5	37.8	68.7	50.6	73.8	63.3	73.8

Table 8: In-sample $\mathbf{R}^2$ (in %) for EDHEC indices (2000-2015)

Strategy	TRP		ARP		SPX + ARP		TRP + ARP	
	5F	10F	5F	10F	5F	10F	5F	10F
CA	58.9	63.1	49.3	61.0	49.3	61.0	59.7	70.8
CTA	13.3	18.4	54.4	62.8	54.5	63.6	54.5	63.6
DS	61.0	64.2	42.6	53.8	48.6	61.3	61.0	66.4
ED	73.7	77.3	42.1	66.6	51.8	71.2	73.7	77.9
EM	87.4	87.8	62.9	77.5	71.2	80.7	87.4	88.6
EMN	33.0	37.0	24.2	46.9	31.4	46.9	33.9	46.9
FIA	57.6	64.4	54.3	60.3	54.3	60.3	61.7	73.4
GM	44.8	53.3	39.0	54.5	39.0	62.4	51.9	62.7
LSE	81.5	84.8	46.9	76.2	80.8	87.9	81.5	87.9
MA	46.7	50.1	24.0	50.0	39.6	62.4	46.7	64.4
RV	74.8	79.2	56.0	74.8	66.5	78.4	74.8	82.0
SB	78.9	81.0	59.2	71.6	86.2	89.1	86.2	89.1
FOF	62.0	66.9	43.5	68.4	53.7	74.1	62.0	74.1

Figure 35: Number of alternative and traditional selected risk factors (HFR)

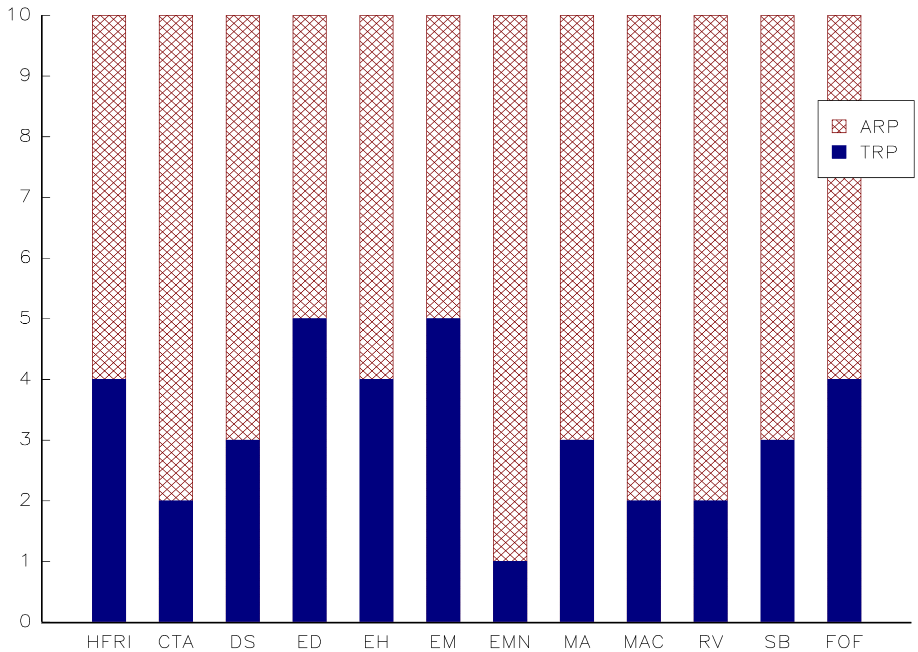
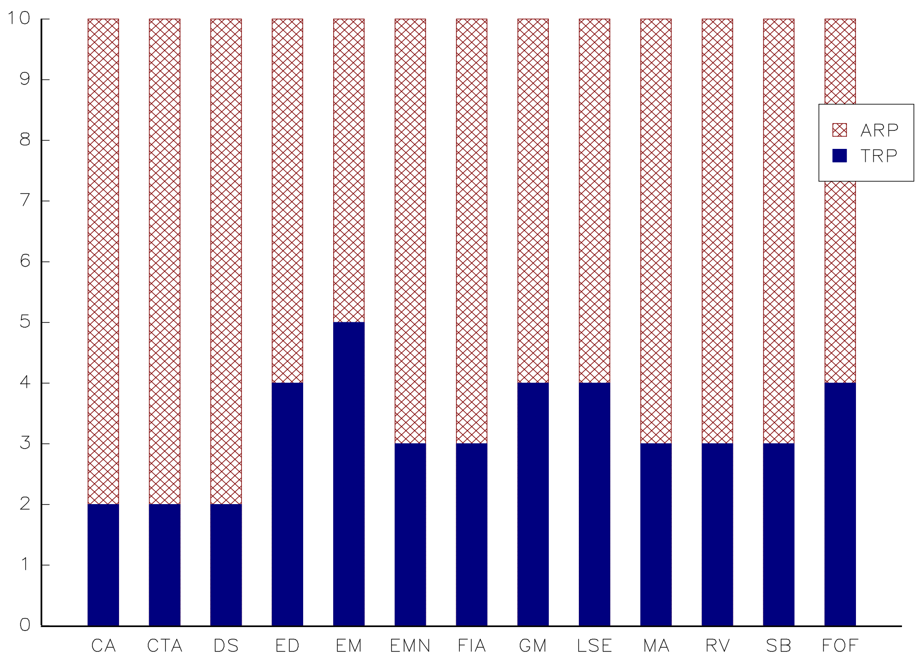


Figure 36: Number of alternative and traditional selected risk factors (EDHEC)



5.3 Dynamic out-of-sample analysis

In the paragraph above, the results are obtained using an in-sample procedure meaning that the exposure $\hat{\beta}_{i,t}^{j}$ at time t is estimated using future information. Indeed, the estimated values $\hat{\beta}_{i,t}^{j}$ reflect all the information of the sample between $t = 1$ and $t = T$. In what follows, we consider an out-of-sample procedure:

- The exposure $\hat{\beta}_{i,t}^{j}$ is estimated by using the information included in the sample until $t - 1$.
- These estimated values are valid for the time period t, meaning that the monthly return forecasted by the model is:

$$\hat{R}_{i,t} = R_{f,t} + \sum_{j=1}^{n_{\mathcal{F}}} \hat{\beta}_{i,t}^{j} \mathcal{F}_{j,t}$$

The procedure described above is the core of hedge fund replication and is extensively used to build hedge fund clones. In our case, the estimated values $\hat{\beta}_{i,t}^{j}$ are based on the lasso approach using a 24-month rolling window $[t - 24, t - 1]$ and 10 selected risk factors. In order to obtain the statistics for the full period (2000-2015), we consider the data from December 1997. In Tables 9 and 10, we report the correlation ρ_i and the tracking error TE_i between the hedge fund index i and the corresponding hedge fund clone[53]. We notice that the improvement due to alternative risk premia is not very significant in terms of correlation, except for CTA and merger arbitrage strategies. The reason for this is that the correlation is already high when we consider traditional risk premia. However, we obtain better results in terms of tracking error when we consider alternative risk factors.

Table 9: Out-of-sample correlation and tracking error (in %) for HFR indices (2000-2015)

Strategy	Correlation				Tracking error			
	TRP	ARP	SPX + ARP	TRP + ARP	TRP	ARP	SPX + ARP	TRP + ARP
HFRI	89.4	82.3	88.9	87.2	3.2	3.8	3.0	3.3
CTA	34.9	54.9	55.3	53.9	9.1	8.0	7.8	7.8
DS	61.4	45.3	48.2	65.4	5.5	6.1	6.2	5.0
ED	81.1	69.1	76.9	81.7	4.1	4.9	4.2	3.7
EH	89.8	83.7	89.3	90.0	4.1	5.0	4.0	3.9
EM	85.9	69.6	71.6	87.9	6.4	8.9	8.5	5.7
EMN	39.8	56.3	59.0	59.0	3.3	2.7	2.7	2.6
MA	58.6	61.9	63.4	63.9	3.3	3.0	2.8	2.8
MAC	55.0	65.7	65.5	65.4	5.4	4.5	4.4	4.7
RV	77.8	58.4	62.5	73.8	2.9	3.6	3.4	2.9
SB	81.6	81.6	88.2	88.9	10.1	9.8	7.8	7.6
FOF	75.6	74.2	76.9	77.8	4.2	4.1	3.9	3.8

We can also compute the performance ratio as the cumulative return of the clone divided by the cumulative return of the index:

$$\mathfrak{R}_i = \prod_{t=1}^{T} \left(1 + \hat{R}_{i,t}\right) \Big/ \prod_{t=1}^{T} \left(1 + R_{i,t}\right)$$

[53] We have $\rho_i = \mathrm{cor}\left(R_{i,t}, \hat{R}_{i,t}\right)$ and $\mathrm{TE}_i = \sqrt{12} \times \sigma\left(R_{i,t} - \hat{R}_{i,t}\right)$.

Table 10: Out-of-sample correlation and tracking error (in %) for EDHEC indices (2000-2015)

Strategy	Correlation				Tracking error			
	TRP	ARP	SPX + ARP	TRP + ARP	TRP	ARP	SPX + ARP	TRP + ARP
CA	66.0	60.9	58.3	61.7	5.3	5.4	5.6	5.4
CTA	27.0	59.0	59.3	57.1	10.0	7.9	7.8	7.8
DS	66.7	55.2	61.2	65.6	4.9	5.4	5.0	4.8
ED	77.7	68.5	73.4	81.5	4.0	4.5	4.0	3.4
EM	86.3	73.4	76.4	86.7	5.5	7.0	6.7	5.3
EMN	54.5	42.0	44.2	43.8	2.8	3.1	3.0	3.0
FIA	63.8	46.8	47.4	51.6	3.5	3.6	3.6	3.4
GM	65.6	64.6	64.5	66.7	4.2	3.9	3.9	4.0
LSE	86.9	80.2	87.2	86.8	3.7	4.4	3.5	3.6
MA	56.0	57.5	53.8	59.2	3.7	2.9	3.1	2.9
RV	78.3	68.8	71.9	76.2	2.9	3.2	3.1	2.8
SB	86.0	77.9	90.8	91.3	8.4	10.6	6.4	6.4
FOF	76.6	75.7	77.0	81.2	3.9	4.0	4.0	3.5

$\mathfrak{R}_i$ indicates the proportion of the hedge fund's performance that can be replicated using risk premia, whereas $(1 - \mathfrak{R}_i)$ is the alpha that is not due to time-varying exposure to traditional and alternative risk premia. Results are reported in Appendix in Tables 16 and 17. It is interesting to note that using ARP instead of TRP significantly improves the performance of hedge fund clones. For instance, by using TRP, we replicate 72% of the HFRI performance whereas we replicate 105% when we consider SPX and ARP.

We now study four emblematic hedge fund strategies: CTA, long/short equity (or EH), merger arbitrage and relative value. For each strategy, we calculate the frequencies of non-zero exposure estimated previously with the 10-factor lasso model. In Tables 11 and 12, we report the 10 most frequent risk premia and the corresponding frequencies[54] (in %) for the four hedge fund strategies. We observe some interesting facts:

- For the CTA strategy, the most important risk factors are momentum risk premia, especially in commodities, currencies and rates.
- We verify that returns on long/short equity strategies are explained by equity risk factors. However, we notice that they also incorporate a currencies/carry risk premium.
- The main important risk factor of the merger arbitrage strategy is the merger arbitrage risk premium.
- The risk factors of the relative value strategy are carry risk premia (HY, short volatility and currencies).

What is remarkable is the strong consistency between HFR and EDHEC indices. Indeed, they are generally explained by the same risk factors even if the calibration set consists of 71 risk premia.

[54]For instance, a frequency of 50% indicates that the risk premium has been selected half the time, i.e. in alternate months.

Table 11: The 10 most frequent risk premia for HFR indices (2000-2015)

Strategy	Frequency	Risk premia
CTA	56.0	SPX
	44.5	commodities/momentum/cross-section
	42.4	commodities/momentum/time-series
	40.8	equities/growth/US
	37.7	currencies/momentum/time-series/DM
	35.1	currencies/momentum/time-series/EM
	27.2	rates/momentum/time-series/DM
	25.1	equities/low volatility/Japan
	24.6	equities/event/merger arbitrage/DM
	24.6	equities/value/US
EH	100.0	SPX
	77.0	equities/growth/US
	55.5	HY
	50.3	equities/volatility/carry/US
	46.1	MXEF
	46.1	equities/low volatility/EM
	39.8	equities/low volatility/Asia Pacific
	36.6	equities/low volatility/US
	35.1	RTY
	34.6	equities/event/merger arbitrage/DM
MA	88.0	equities/event/merger arbitrage/DM
	65.4	SPX
	62.3	HY
	51.8	equities/volatility/carry/US
	38.7	equities/quality/Europe
	28.3	equities/momentum/cross-section/Europe
	27.7	RTY
	27.7	equities/volatility/carry/Europe
	26.2	equities/reversal/time-series/US
	25.1	equities/low volatility/EM
RV	81.7	HY
	67.5	equities/volatility/carry/US
	55.0	equities/event/merger arbitrage/DM
	39.3	currencies/carry/FRB/DM
	37.7	currencies/carry/FRB/EM
	37.2	equities/momentum/cross-section/Europe
	33.0	SPX
	29.3	equities/value/DM
	27.7	EMBI
	25.7	commodities/carry/TSS

Table 12: The 10 most frequent risk premia for EDHEC indices (2000-2015)

Strategy	Frequency	Risk premia
CTA	84.8	currencies/momentum/time-series/DM
	72.8	commodities/momentum/time-series
	72.8	rates/momentum/time-series/DM
	48.2	commodities/momentum/cross-section
	38.7	currencies/momentum/time-series/EM
	31.9	GSCI
	29.3	GOLD
	24.6	equities/growth/Japan
	23.0	commodities/carry/TSS
	23.0	commodities/liquidity
LSE	98.4	SPX
	75.4	equities/growth/US
	57.1	equities/volatility/carry/US
	53.9	HY
	44.0	equities/low volatility/Asia Pacific
	42.4	currencies/carry/FRB/EM
	37.2	equities/event/merger arbitrage/DM
	36.6	MXEF
	33.5	equities/momentum/cross-section/US
	31.4	equities/low volatility/EM
MA	89.5	equities/event/merger arbitrage/DM
	72.3	HY
	58.6	SPX
	51.8	equities/volatility/carry/US
	40.8	equities/quality/Europe
	34.0	equities/growth/US
	33.0	equities/volatility/carry/Europe
	30.4	equities/momentum/cross-section/Europe
	27.7	equities/reversal/time-series/US
	20.9	EMBI
RV	79.6	HY
	67.5	SPX
	66.5	equities/volatility/carry/US
	63.4	equities/event/merger arbitrage/DM
	40.3	currencies/carry/FRB/DM
	36.1	equities/quality/Europe
	29.8	currencies/carry/FRB/EM
	28.8	equities/growth/US
	28.8	equities/growth/Europe
	27.7	equities/value/DM

On pages 110 to 113, we conduct the same analysis by making a distinction before and after the financial crisis. We generally observe that the size factor (RTY) is less present after 2008, whereas the credit factor (HY) is more frequent after the financial crisis. If we consider the CTA strategy more specifically, the exposure to momentum risk factors has increased over time. In the case of the long/short equity strategy, the analysis highlights the increasing role of low volatility factors. Concerning the merger arbitrage strategy, we do not find a real difference between the two periods. This is not the case for the relative value strategy, which is highly more exposed to the short volatility risk factor after 2009.

Summary 21 *The out-of-sample analysis shows that hedge fund clones with alternative risk premia have the same correlation and tracking error as hedge fund clones with traditional risk premia, but a better performance. Among the various alternative risk premia, the main pertinent risk factors are momentum (commodities and currencies) short volatility (equities and rates) and low volatility (equities).*

6 Conclusion

The literature on alternative risk premia is fragmented and generally centered on exploring backtests of strategies. In particular, many professional studies are examples of "*the blue line is above the red line*" syndrome. By focusing on performance, we have acquired a little insight into the behavior of these alternative risk premia. In this study, we have tried to go beyond the return consideration in order to better understand the important mechanisms behind alternative risk premia.

There is confusion with regard to risk premia, risk factors and market anomalies. A risk premium is compensation for being exposed to a non-diversifiable risk, risk factors are the systematic components that explain the return variation of diversified portfolios, and a market anomaly is a strategy that exhibits a positive excess return that cannot be explained by a risk premium. For instance, a typical market anomaly is a trend-following strategy. Risk premia and market anomalies are generally risk factors. Some exceptions exist, such as the cat bond risk premium, which is not considered a risk factor. We generally define alternative risk premia in opposition to traditional risk premia, the latter denotes long-only exposure to equities and bonds. For instance, value and short volatility strategies are typical alternative risk premia.

In the consumption-based model, there is a strong connection between (traditional and alternative) risk premia. They reward the risk taken by an investor in bad times. This implies that the drawdowns of risk premia must be positively correlated to bad times. For instance, if we consider the four equity option profiles (long call, short call, long put and short put), only the short put option profile must be considered a risk premium. One consequence is that alternative risk premia exhibit negative skewness and large drawdown with respect to the volatility risk measure. This is why we prefer to speak about skewness risk premia instead of alternative risk premia and to reserve the expression "*alternative risk premia*" to the set of risk factors composed of skewness risk premia and market anomalies. However, we recognize that this market practice is confusing, because it gives the false impression that alternative risk premia exhibit homogenous patterns. This is not true, and market anomalies may exhibit behavior that is different to that of skewness risk premia.

Statistical analysis of alternative risk premia is a daunting challenge. In the case of traditional risk premia, it is relatively easy to study their behavior, because indices exist and have been live for a long time. For instance, the US equity risk premium can be analyzed using the S&P 500 index, the MSCI USA index or the Russell 1000 index. Even if they are calculated by three different calculation agents, the three indices are highly correlated. Therefore, the choice of index is not important when assessing the US equity risk premium. In the case of alternative risk premia, there are many products and indices that can be a candidate for representing a specific risk premium. For instance, there are more than 30 indices in the financial industry that are exposed to the US value equity strategy. In order to avoid selection bias, most academic and professional studies choose to create their own backtest. The problem is that these backtests do not accurately represent an investable portfolio, because they do not generally consider liquidity, capacity and trading issues[55]. This is why we prefer to conduct our analysis by using existing indices. However, contrary to traditional risk premia, these indices are not necessarily highly correlated and they exhibit heterogenous behavior. The reason for this is that there are many ways to implement alternative risk premia. Moreover, some of these products add trading strategies in order to limit the drawdown, improve the Sharpe ratio or combine different risk premia. An example is the combination of quality and carry risk factors in equity indices. Another example involves the short volatility strategy. In some indices, the allocation is time-varying, a stop loss based on the VIX level may be added or the index may be long volatility in certain environments. To avoid these biases and to obtain "*pure*" alternative risk premia, we have developed a statistical algorithm, which consists in eliminating indices that do not share the same patterns among the universe of potential candidates. The generic performance of a given risk premium is then the average performance of the selected indices. This procedure demonstrated the extreme heterogeneity of the performance for some categories and emphasized the important role of due diligence in alternative risk premia. Another lesson concerns the existence of some alternative risk premia. Indeed, an analysis of our database and the results of the algorithm shows that some alternative risk premia do not exist or are not implemented using indices, such as commodities/reversal, credit/momentum, currencies/volatility/carry, equities/liquidity strategies. These results contrast with marketing materials produced by some banks and asset managers, which argue that they can capture these risk premia.

As expected, generic indices of alternative risk premia exhibit negative skewness and some of them present more relative extreme risk than traditional risk premia. This is particularly true for equities/volatility/carry and currencies/carry strategies. We also notice that alternative risk premia improve the volatility diversification of traditional risk premia. However, it is not obvious that they limit tail risk. The issue comes from the skewness aggregation problem. We show that there is a floor to the hedging of the third moment, which is not the case for the second moment. As a consequence, volatility diversification or negative correlation leads to reduced volatility, but increased tail risks in relative terms. This result reinforces the need to distinguish between skewness risk premia and market anomalies. Therefore, we estimate the payoff function associated with each risk premium. To this end, we develop a non-linear estimation based on copulas and quantile regression. We find that some alternative risk premia are independent from traditional risk premia, for instance those based on commodities. Some of them are short put options with respect to the equity risk premium, such as the short volatility strategy, whereas others are leverage portfolios or straddle options. This implies that allocation models based on the volatility risk measure (Markowitz, risk parity, minimum variance, etc.) are not adequate ways to manage a portfolio of alternative risk premia. Indeed, these methods do not account for the

[55]A famous example is the Fama-French selection of European risk factors, based on a universe of 4 700 stocks on average by month.

issue of skewness aggregation and non-linear payoffs, and the strong heterogeneity between skewness risk premia and market anomalies.

The development of alternative risk premia must affect the analysis of hedge fund strategies. Indeed, it is obvious that the boundaries between these two concepts are blurred. For instance, we may wonder whether an equity short volatility or a currency carry exposure is a hedge fund strategy or an alternative risk premium. In practice, we observe significant differences between alternative risk premia and hedge fund strategies, because of issues surrounding implementation. We recall that alternative risk premia are invested using indices. The advantage of these indices is their great transparency, but they are also purely systematic and static. Conversely, hedge fund strategies are less transparent, but we expect them to be actively managed. However, the emergence of alternative risk premia recasts the analysis of hedge fund returns. Until recently, it was acceptable to break hedge fund returns down into three components: traditional beta, dynamic beta and alpha. The traditional beta is the return component explained by a static portfolio of traditional risk premia whereas the alternative beta is the return component explained by a dynamic allocation of traditional risk premia. In this framework, alpha is the unexplained component of hedge fund returns. By including alternative risk premia in the set of risk factors, we can explore new frontiers in the risk/return analysis of the hedge fund industry. For instance, we can estimate the proportion of the performance explained by a static portfolio of alternative risk factors, what traditional risk factors are substituted by alternative risk factors, how much exposure to particular emblematic risk factors (carry trade, short volatility) has evolved over the past 15 years, etc.

Our analysis shows that the main drivers of the hedge fund industry are long exposure on DM equities, long exposure on high yield credit and a subset of alternative risk premia. From a general point of view, we also observe that the size and value equity risk factors, which were ubiquitous at the beginning of the 2000s, have lost much of their significance over recent years. Conversely, the low volatility, momentum and quality equity risk factors increased in importance. We also notice that the risk/return patterns of some hedge fund strategies are difficult to explain through traditional and alternative risk factors (CTA, DS, EMN and GM). In fact, we observe some significant differences between strategies before and after the financial crisis. Another important result is the predominance of commodity and currency risk factors over rates risk factors. Finally, our results support the idea that using alternative risk premia should also be an integral part of the bottom-up approach when selecting hedge fund managers.

The primary lesson of our research concerns the multi-asset allocation design. The development of alternative risk premia extends the universe of risk premia that can be harvested. With the inclusion of these new assets, managing a diversified portfolio cannot be reduced to allocation decisions concerning only the relative weighting of equities and bonds. In this new framework, the choice of whether or not to invest in certain alternative risk premia, the allocation process between skewness risk premia and market anomalies, the use of a loss risk measure instead of the volatility measure and the management of long/short exposure make the allocation process much more difficult to define[56]. The second lesson has consequences for the ongoing debate about passive versus active management. Alternative risk premia explain a proportion of active management performance. Therefore, risk factor analysis can help us to better understand the strategies implemented by active managers.

[56]However, Maeso and Martellini (2016) found that heuristic allocation strategies applied to alternative risk factors could be a good solution for harvesting alternative risk premia.

However, the choice of whether to implement a risk premium or a portfolio of risk premia using active management or an index is not a simple decision. The main reason for this is that the two investments are not perfectly substitutable and their similarity depends on the nature of the risk premium. For instance, investors may prefer a fund manager to invest in long/short equity risk premia, but they may be indifferent between a fund manager and an index manager when it comes to investing in long-only equity risk premia. An active manager may allocate assets between several alternative risk premia, create alpha with respect to the benchmark or manage liquidity and capacity constraints with greater ease. In this context, there is a gap between a risk factor analysis of a given portfolio and the day-to-day management of a risk premia portfolio. Therefore, the debate about passive versus active management will continue. What cannot be disputed is that the emergence of alternative risk premia shall renew the benchmarking issue and the risk/return analysis of active management.

References

[1] Amenc, N., Martellini, L., Meyfredi, J. C., and Ziemann, V. (2010), Passive Hedge Fund Replication – Beyond the Linear Case, *European Financial Management*, 16(2), pp. 191-210.

[2] Amihud, Y. (2002), Illiquidity and Stock Returns: Cross-Section and Time-Series Effects, *Journal of Financial Markets*, 5(1), pp. 31-56.

[3] Ang, A. (2014), *Asset Management – A Systematic Approach to Factor Investing*, Oxford University Press.

[4] Asness, C.S., and Frazzini, A. (2013), The Devil in HML's details, *Journal of Portfolio Management*, 39(4), pp. 49-68.

[5] Asness, C.S., Moskowitz, T.J., and Pedersen, L.H. (2013), Value and Momentum Everywhere, *Journal of Finance*, 68(3), pp. 929-985.

[6] Avramov, D., Chordia, T., and Goyal, A. (2006), Liquidity and Autocorrelations in Individual Stock Returns, *Journal of Finance*, 61(5), pp. 2365-2394.

[7] Bailey, D.H., Borwein, J.M., Lopez de Prado, M., and Zhu, Q.J. (2016), The Probability of Backtest Overfitting, *Journal of Computational Finance*, forthcoming.

[8] Baker, M., Bradley, B., and Taliaferro, R. (2014), The Low-Risk Anomaly: A Decomposition into Micro and Macro Effects, *Financial Analysts Journal*, 70(2), pp. 43-58.

[9] Banz, R.W. (1981), The Relationship between Return and Market Value of Common Stocks, *Journal of Financial Economics*, 9(1), pp. 3-18.

[10] Barberis, N., Shleifer, A., and Vishny, R. (1998), A Model of Investor Sentiment, *Journal of Financial Economics*, 49(3), pp. 307-343.

[11] Barberis, N., and Thaler, R.H. (2003), A Survey of Behavioral Finance, in Constantinides, G.M., Harris, M. and Stulz, R.M. (Eds), *Handbook of the Economics of Finance*, 1(B), Elsevier, pp. 1053-1128.

[12] Basu, S. (1977), Investment Performance of Common Stocks in Relation to Their Price-Earnings Ratios: A Test of The Efficient Market Hypothesis, *Journal of Finance*, 32(3), pp. 663-682.

[13] Black, F. (1972), Capital Market Equilibrium with Restricted Borrowing, *Journal of Business*, 45(3), pp. 444-455.

[14] Briere, M., Burgues, A., and Signori, O. (2010), Volatility Exposure for Strategic Asset Allocation, *Journal of Portfolio Management*, 36(3), pp. 105-116.

[15] Broadie, M., Chernov, M., and Johannes, M. (2009), Understanding Index Option Returns, *Review of Financial Studies*, 22(11), pp. 4493-4529.

[16] Bruder, B., Dao, T.L., Richard, J.C., and Roncalli, T. (2011), Trend Filtering Methods for Momentum Strategies, *SSRN*, `www.ssrn.com/abstract=2289097`.

[17] Bruder, B., and Gaussel, N. (2011), Risk-Return Analysis of Dynamic Investment Strategies, *SSRN*, `www.ssrn.com/abstract=2465623`.

[18] Bruder, B., Gaussel, N., Richard, J.C., and Roncalli (2013), T., Regularization of Portfolio Allocation, *SSRN*, `www.ssrn.com/abstract=2767358`.

[19] Carhart, M.M. (1997), On Persistence in Mutual Fund Performance, *Journal of Finance*, 52(1), pp. 57-82.

[20] Carr, P., and Wu, L. (2009), Variance Risk Premiums, *Review of Financial Studies*, 22(3), pp. 1311-1341.

[21] Cazalet, Z., and Roncalli, T. (2014), Facts and Fantasies About Factor Investing, *SSRN*, `www.ssrn.com/abstract=2524547`.

[22] Cochrane, J.H. (2001), *Asset Pricing*, Princeton University Press.

[23] Cochrane, J.H. (2011), Presidential Address: Discount Rates, *Journal of Finance*, 66(4), pp. 1047-1108.

[24] Correia, M., Richardson, S., and Tuna, I. (2012), Value Investing in Credit Markets, *Review of Accounting Studies*, 17(3), pp. 572-609.

[25] De Bondt, W.F., and Thaler, R. (1985), Does The Stock Market Overreact?, *Journal of Finance*, 40(3), pp. 793-805.

[26] El Karoui, N., Jeanblanc, M., and Shreve, S.E. (1998), Robustness of the Black and Scholes Formula, *Mathematical Finance*, 8(2), pp. 93-126.

[27] Erb, C.B., and Harvey, C.R. (2006), The Strategic and Tactical Value of Commodity Futures, *Financial Analysts Journal*, 62(2), pp. 69-97.

[28] Fama, E.F., and French, K.R. (1992), The Cross-Section of Expected Stock Returns, *Journal of Finance*, 47(2), pp. 427-465.

[29] Fama, E.F., and French, K.R. (1993), Common Risk Factors in the Returns on Stocks and Bonds, *Journal of Financial Economics*, 33(1), pp. 3-56.

[30] Fama, E.F., and French, K.R. (1998), Value versus Growth: The International Evidence, *Journal of Finance*, 53(6), pp. 1975-1999.

[31] Fama, E.F., and French, K.R. (2012), Size, Value, and Momentum in International Stock Returns, *Journal of Financial Economics*, 105(3), pp. 457-472.

[32] Fama, E.F., and MacBeth, J.D. (1973), Risk, Return, and Equilibrium: Empirical Tests, *Journal of Political Economy*, 81(3), pp. 607-636.

[33] Frazzini, A., and Pedersen, L.H. (2014), Betting Against Beta, *Journal of Financial Economics*, 111(1), pp. 1-25.

[34] Fung, W., and Hsieh, D.A. (1997), Empirical Characteristics of Dynamic Trading Strategies: the Case of Hedge Funds, *Review of Financial Studies*, 10(2), pp. 275-302.

[35] Fung, W., and Hsieh, D.A. (2001), The Risk in Hedge Fund Strategies: Theory and Evidence from Trend Followers, *Review of Financial studies*, 14(2), pp. 313-341.

[36] Giamouridis, D., and Paterlini, S. (2010), Regular(ized) Hedge Fund Clones, *Journal of Financial Research*, 33(3), pp. 223-247.

[37] Gibbons, M.R., Ross, S.A., and Shanken, J. (1989), A Test of the Efficiency of a Given Portfolio, *Econometrica*, 57(5), pp. 1121-1152.

[38] Goetzmann, W., Ingersoll, J., Spiegel, M.I., and Welch, I. (2002), Sharpening Sharpe Ratios, *National Bureau of Economic Research*, 9116.

[39] Grebenkov, D.S., and Serror, J. (2014), Following a Trend with an Exponential Moving Average: Analytical Results for a Gaussian Model, *Physica A: Statistical Mechanics and its Applications*, 394, pp. 288-303.

[40] Grinblatt, M., Titman, S., and Wermers, R. (1995), Momentum Investment Strategies, Portfolio Performance, and Herding: A Study of Mutual Fund Behavior, *American Economic Review*, 85(5), pp. 1088-1105.

[41] Harvey, C.R., Liu, Y., and Zhu, H. (2016), ...and the Cross-Section of Expected Returns, *Review of Financial Studies*, 29(1), pp. 5-68.

[42] Hasanhodzic, J., and Lo, A.W. (2007), Can Hedge-Fund Returns Be Replicated?: The Linear Case, *Journal of Investment Management*, 5(2), pp. 5-45.

[43] Hirshleifer, D. (2001), Investor Psychology and Asset Pricing, *Journal of Finance*, 56(4), pp. 1533-1597.

[44] Hou, K., Xue, C., and Zhang, L. (2015), Digesting Anomalies: An Investment Approach, *Review of Financial Studies*, 28(3), pp. 650-705.

[45] Houweling, P., and van Zundert, J. (2015), Factor Investing in the Corporate Bond Market, *SSRN*, `www.ssrn.com/abstract=2516322`.

[46] Ilmanen, A. (2011), *Expected Returns: An Investor's Guide to Harvesting Market Rewards*, Wiley.

[47] Jegadeesh, N. (1990), Evidence of Predictable Behavior of Security Returns, *Journal of Finance*, 45(3), pp. 881-898.

[48] Jegadeesh, N., and Titman, S. (1993), Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency, *Journal of Finance*, 48(1), pp. 65-91.

[49] Jurczenko, E., and Maillet, B. (2006), Multi-moment Asset Allocation and Pricing Models, John Wiley & Sons.

[50] Koijen, R.S.J., Moskowitz, T.J., Pedersen, L.H., and Vrugt, E.B. (2015), Carry, *SSRN*, `www.ssrn.com/abstract=2298565`.

[51] Lehmann, B.N. (1990), Fads, Martingales and Market Efficiency, *Quarterly Journal of Economics*, 105(1), pp. 1-28.

[52] Leland, H.E. (1999), Beyond Mean-Variance: Performance Measurement in a Nonsymmetrical World, *Financial Analysts Journal*, 55(1), pp. 27-36.

[53] Lempérière, Y., Deremble, C., Nguyen, T.T., Seager, P., Potters, M., and Bouchaud, J-P. (2014a), Risk Premia: Asymmetric Tail Risks and Excess Returns, *SSRN*, `www.ssrn.com/abstract=2502743`.

[54] Lempérière, Y., Deremble, C., Seager, P., Potters, M., and Bouchaud, J-P. (2014b), Two Centuries of Trend Following, *Journal of Investment Strategies*, 3(3), pp. 41-61.

[55] Lo A.W., and MacKinlay, A.C. (1990), When are Contrarian Profits due to Stock Market Overreaction?, *Review of Financial Studies*, 3(2), pp. 175-205.

[56] Maeso, J-M. and Martellini, L. (2016), Factor Investing and Risk Allocation: From Traditional to Alternative Risk Premia Harvesting, *EDHEC Working Paper*.

[57] Malamud, S. (2014), Portfolio Selection with Options and Transaction Costs, *Swiss Finance Institute Research Paper*, 14-08.

[58] Markowitz, H. (1952), Portfolio Selection, *Journal of Finance*, 7(1), pp. 77-91.

[59] Martellini, L. and Milhau, V. (2015), Factor Investing: A Welfare Improving New Investment Paradigm or Yet Another Marketing Fad?, *EDHEC Working Paper*.

[60] Martellini, L. and Ziemann, V. (2010), Improved Estimates of Higher-Order Comoments and Implications for Portfolio Selection, *Review of Financial Studies*, 23(4), pp. 1467-1502.

[61] Martin, R.J. (2012), Momentum Trading: 'Skews Me, *Risk*, 25(8), pp. 40-45.

[62] McLean, R.D., and Pontiff, J. (2016), Does Academic Research Destroy Stock Return Predictability?, *Journal of Finance*, 71(1), pp. 5-32.

[63] Mendoza, E.G. (1995), The Terms of Trade, the Real Exchange Rate, and Economic Fluctuations, *International Economic Review*, 36(1), pp. 101-137.

[64] Miffre, J., and Rallis, G. (2007), Momentum Strategies in Commodity Futures Markets, *Journal of Banking & Finance*, 31(6), pp. 1863-1886.

[65] Moskowitz, T.J., Ooi, Y.H., and Pedersen, L.H. (2012), Time Series Momentum, *Journal of Financial Economics*, 104(2), pp. 228-250.

[66] Novy-Marx, R. (2013), The Other Side of Value: The Gross Profitability Premium, *Journal of Financial Economics*, 108(1), pp. 1-28.

[67] Novy-Marx, R. (2014), Predicting Anomaly Performance with Politics, The Weather, Global Warming, Sunspots, and The Stars, *Journal of Financial Economics*, 112(2), pp. 137-146.

[68] Pàstor, L., and Stambaugh, R.F. (2003), Liquidity Risk and Expected Stock Returns, *Journal of Political Economy*, 111(3), pp. 642-685.

[69] Piotroski, J.D. (2000), Value Investing: The Use of Historical Financial Statement Information to Separate Winners from Losers, *Journal of Accounting Research*, 38, pp. 1-41.

[70] Potters, M., and Bouchaud, J-P. (2006), Trend Followers Lose More Often Than They Gain, *Wilmott Magazine*, 26, pp. 58-63.

[71] Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.

[72] Roncalli, T., and Teiletche, J. (2008), An Alternative Approach to Alternative Beta, *Journal of Financial Transformation*, 24, pp. 43-52.

[73] Roncalli, T., and Weisang, G. (2011), Tracking Problems, Hedge Fund Replications and Alternative Beta, *Journal of Financial Transformation*, Cass-Capco Institute Paper Series on Risk, 31, pp. 19-30.

[74] Ross, S. (1976), The Arbitrage Theory of Capital Asset Pricing, *Journal of Economic Theory*, 13(3), pp. 341-360.

[75] Rozeff, M.S., and Kinney, W.R. (1976), Capital Market Seasonality: The Case of Stock Returns, *Journal of Financial Economics*, 3(4), pp. 379-402.

[76] Rouwenhorst, K.G. (1998), International Momentum Strategies, *Journal of Finance*, 53(1), pp. 267-284.

[77] Sharpe, W.F. (1964), Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance*, 19(3), pp. 425-442.

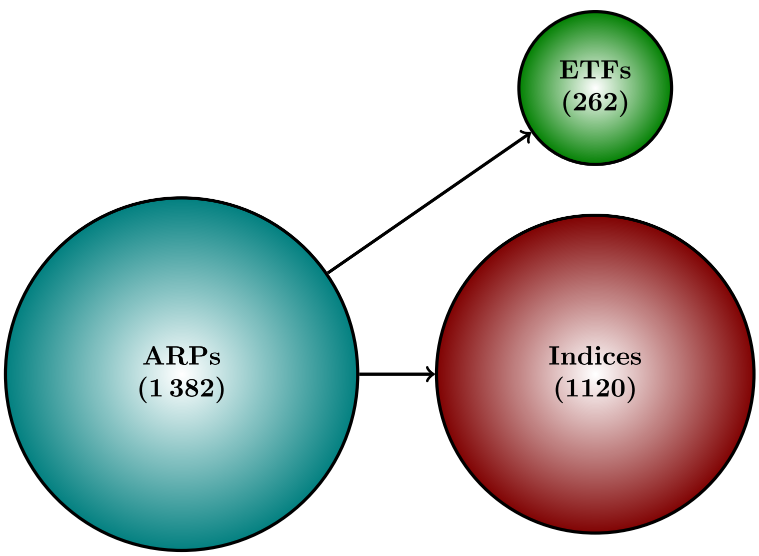
[78] Tibshirani, R. (1996), Regression Shrinkage and Selection via the Lasso, *Journal of the Royal Statistical Society B*, 58(1), pp. 267-288.

[79] Tobin, J. (1958), Liquidity Preference as Behavior Towards Risk, *Review of Economic Studies*, 25(2), pp. 65-86.

A Description of the ARP database

The ARP database contains about 2 000 observations (indices and ETFs) that are presented as alternative risk premia products. However, many of these products cannot be used in our analysis, because they combine different risk premia or they are trading strategies disguised as risk premia. An example is the volatility risk premium, which consists of being long on implied volatility and short on realized volatility. In some products, a stop loss is added depending on the level of the VIX index or the strategy can even go long on realized volatility and short on implied volatility. In this case, it is obvious that the behavior of the strategy cannot be fully explained by the volatility risk premium. Moreover, we only use products that do not represent cap-weighted portfolios. This point is particularly important for the size factor. After cleaning the database, we obtain a universe of 1 382 products.

Figure 37: Breakdown between ETF products and indices



In Figure 37, we indicate the breakdown between ETF products and indices. Indices represent more than 80% of the eligible observations. Moreover, we also notice that all these ETF products replicate an index, meaning that their behavior is already taken into account in the universe of indices. Most of these ETFs correspond to equity risk factors and only 10 ETFs[57] products concern risk factors of the other asset classes. Concerning equity ETFs, the two main represented risk factors are carry and value (see Figure 38). These are followed by low volatility and growth. Quality and momentum remain marginal in the ETF market.

If we analyze the universe of 1 120 eligible indices, 67.5% of them concern equity risk factors (Figure 39). However, this over-representation of equity products can be explained by the typology of index sponsors. Indeed, asset managers and index providers focus almost exclusively on the equity asset class (Table 13). If we limit our analysis to bank indices, we obtain a more balanced picture between equity, fixed-income, foreign exchange and commodity indices.

Figure 38: Breakdown of eligible equity ETFs between risk factors

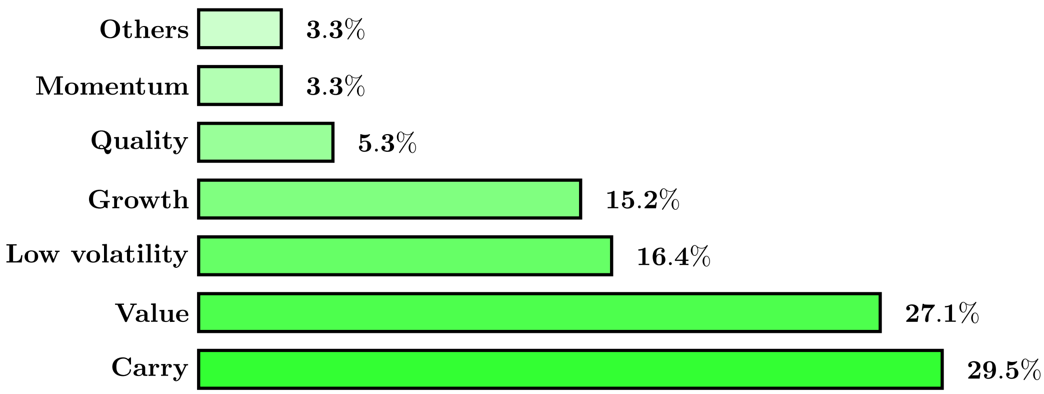


Figure 39: Breakdown of eligible indices between asset classes

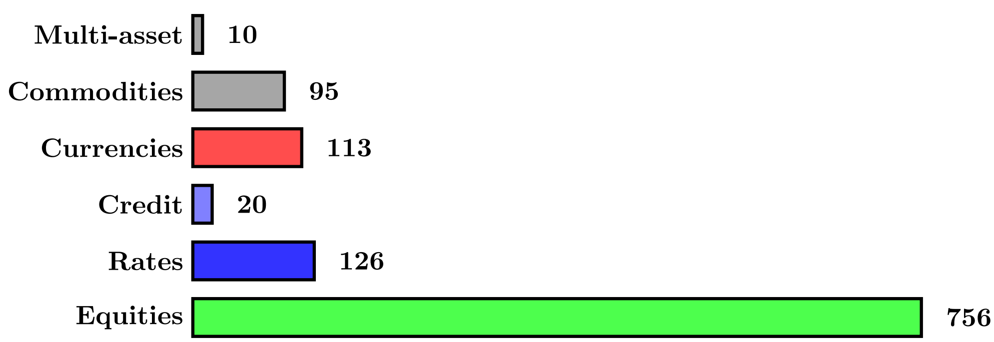


Table 13: Breakdown of risk factors by asset classes and index sponsors

Sponsor	Equities	Rates	Credit	Currencies	Commodities	Multi-Asset
Asset managers	45					
Banks	270	126	20	113	88	7
Index providers	441				7	3
All	756	126	20	113	95	10

We now focus on bank's proprietary indices[58]. In the following pages, we report the graph database of risk factors obtained using the Neo4j software. Each index is represented by a green node. Asset classes, risk factors and regions correspond to yellow, red and violet nodes, respectively.

Figure 40 gives an overview of the mapping between the categories and the asset classes. We notice that some risk factors are better represented than others. This is the case for the momentum risk factor, which is highly connected to the five asset classes. Carry is also well represented and has a high interconnectedness ratio with fixed-income, foreign exchange and commodity asset classes. Bank's proprietary indices on carry are also present in the equity asset class, but not to the same degree. This is explained because banks face a fierce competition from asset managers and index providers, such as Dow Jones, MSCI, S&P and STOXX. The value strategy mainly concerns the equity and foreign exchange asset classes, whereas the volatility risk factor is more present in equities and rates. Five risk factors (event, growth, low volatility, quality and size) exclusively affect equity indices. Finally, liquidity and reversal are the least developed risk factors.

In Figures 41 to 44, we report graph relationships between the different nodes for each asset class. We notice that banks can have more than one index per risk factor and asset class. They generally launch a risk factor across several regions and with different parameter specifications[59]. In order of importance, the most represented regions are:

1. Developed markets (DM).
2. Both developed and emerging markets (Global).
3. US.
4. Europe (mainly the Eurozone).
5. Japan.
6. Emerging markets (EM).

However, we observe some differences between asset classes. Fixed-income risk factors are mainly implemented in DM, US and Europe, whereas foreign exchange risk factors are not implemented at the country level.

[57]They are mainly ETNs.

[58]The universe includes products from Barclays, Credit Suisse, Citibank, Deutsche Bank, HSBC, J.P. Morgan, Merrill Lynch/Bank of America, Morgan Stanley, Nomura, Société Générale and UBS. This is an incomplete list, but it may be extremely difficult to obtain information for other banks.

[59]For instance, the equity volatility premium can be implemented using put/call options, straddle/strangle derivatives or variance/volatility swaps. The choice of maturity is another important parameter. Moreover, similar indices can coexist within the same bank, because they have been developed by different teams.

Figure 40: Graph database of risk factors

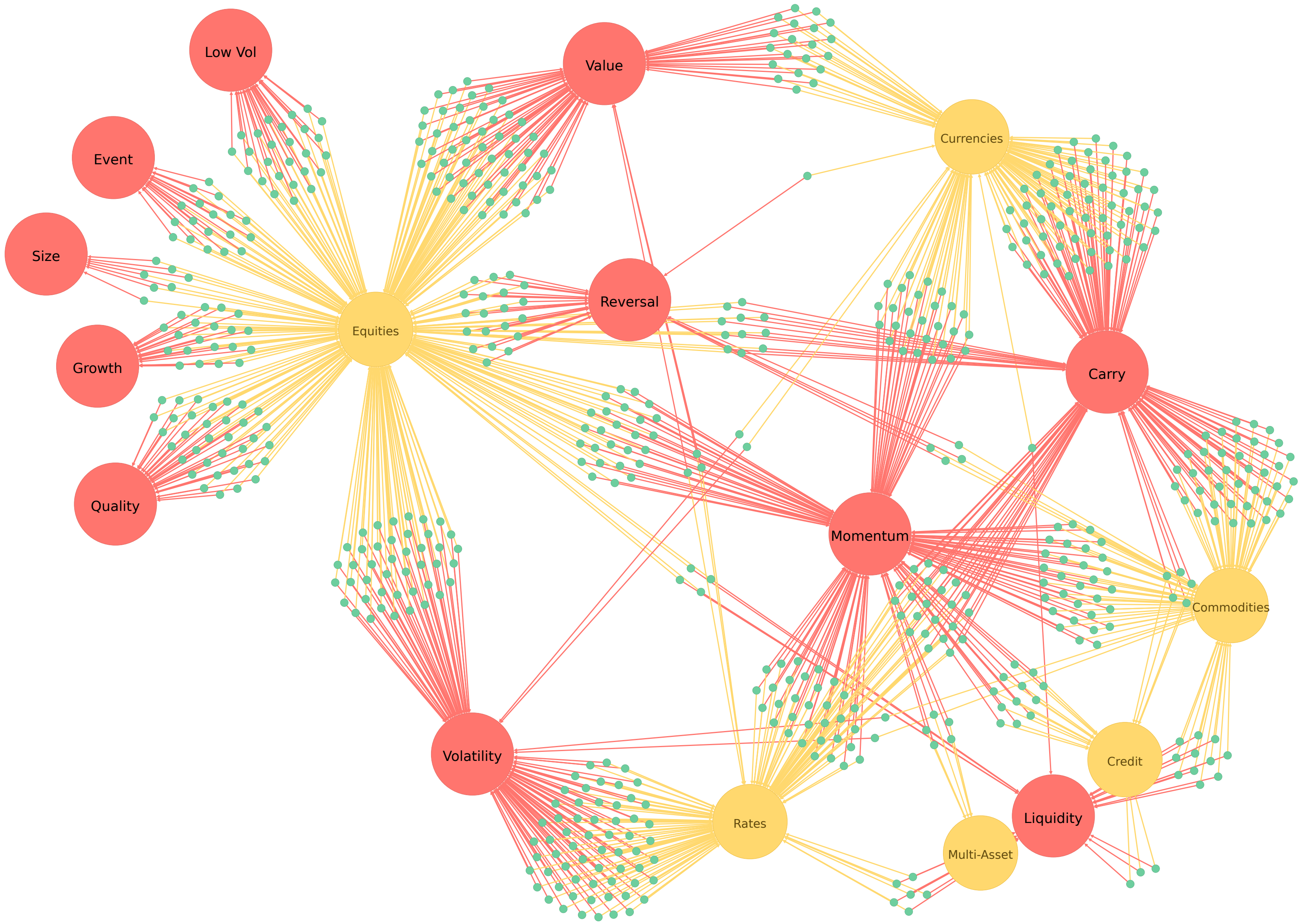


Figure 41: Graph database of equity risk factors

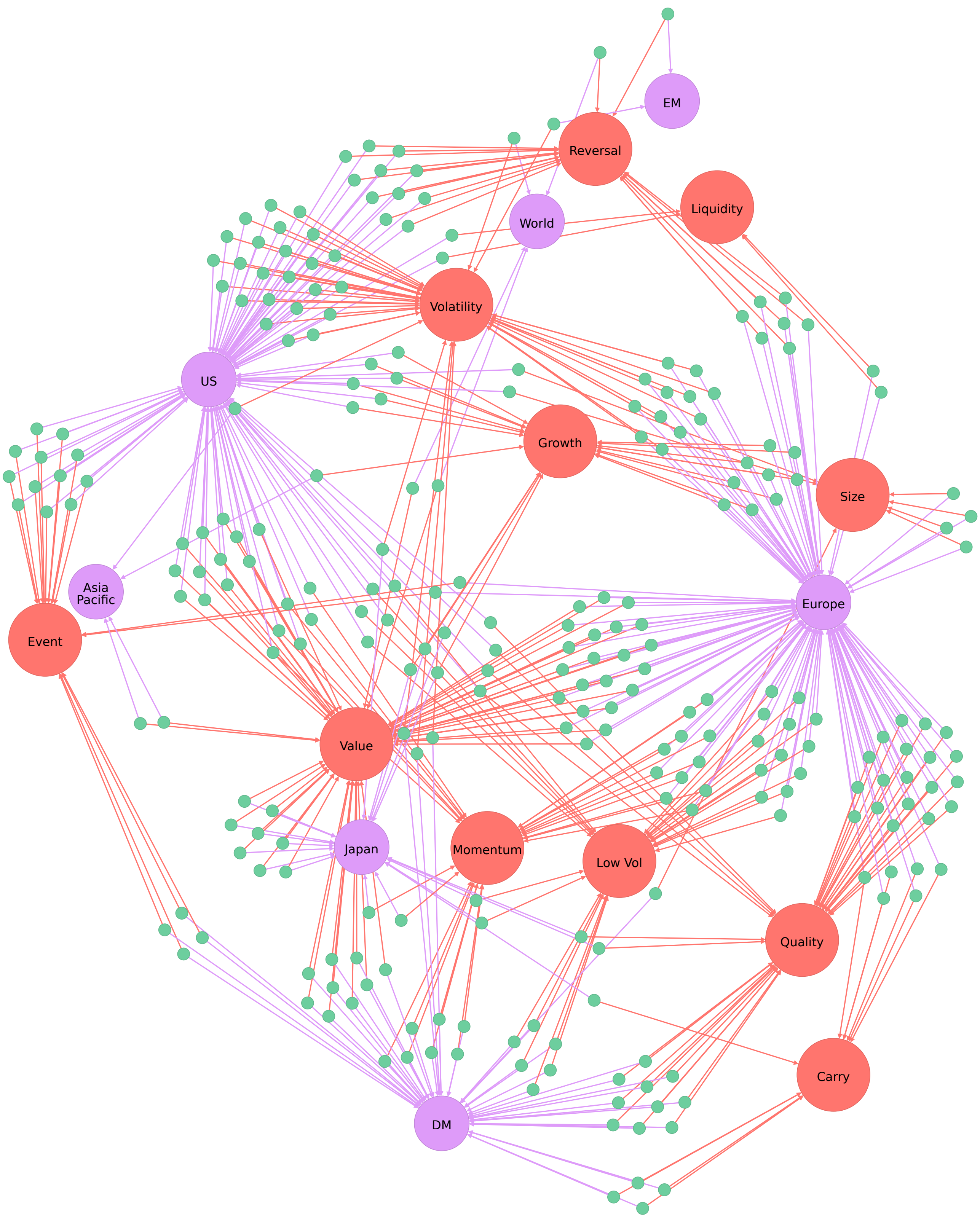


Figure 42: Graph database of fixed-income risk factors

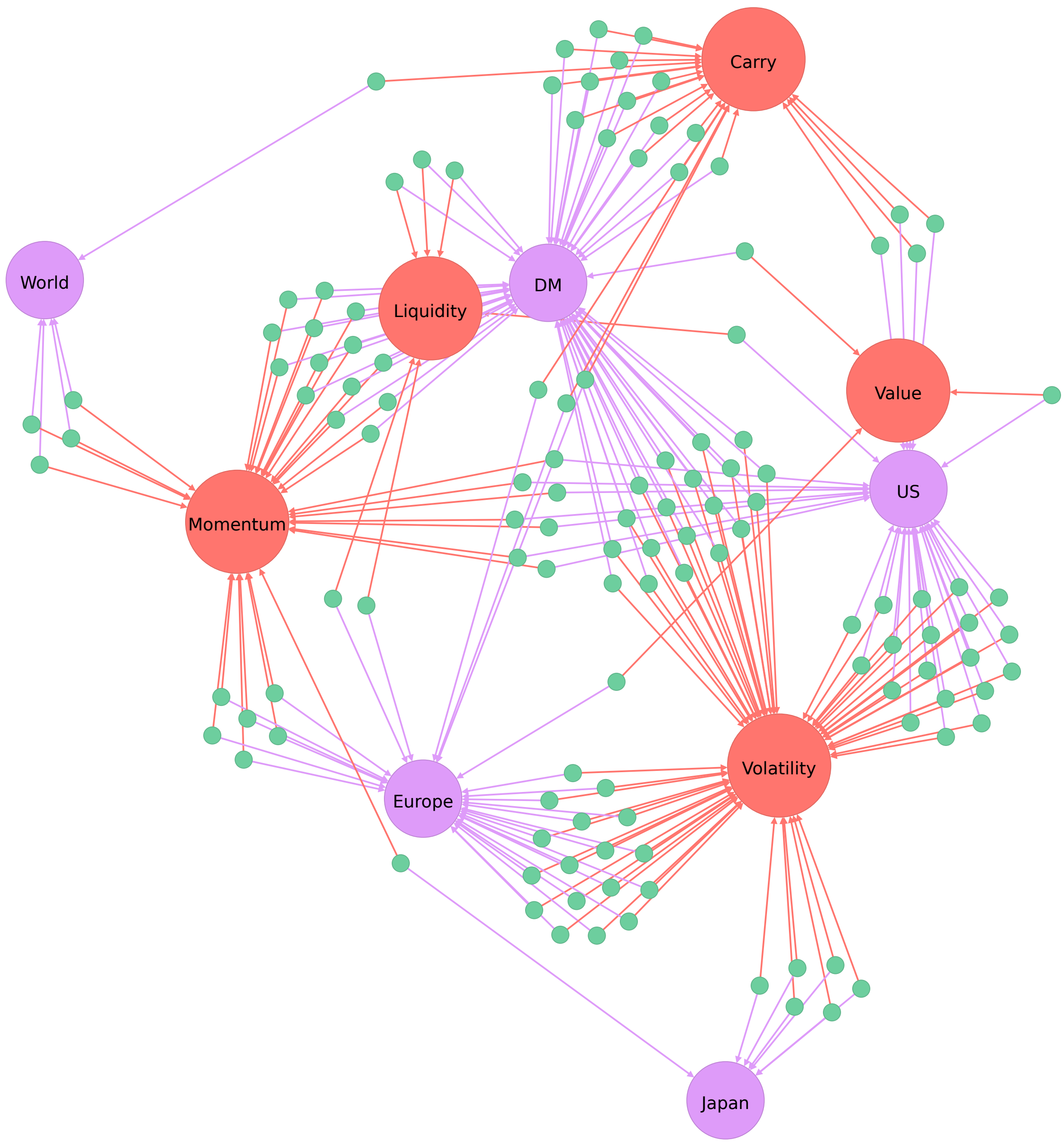


Figure 43: Graph database of credit risk factors

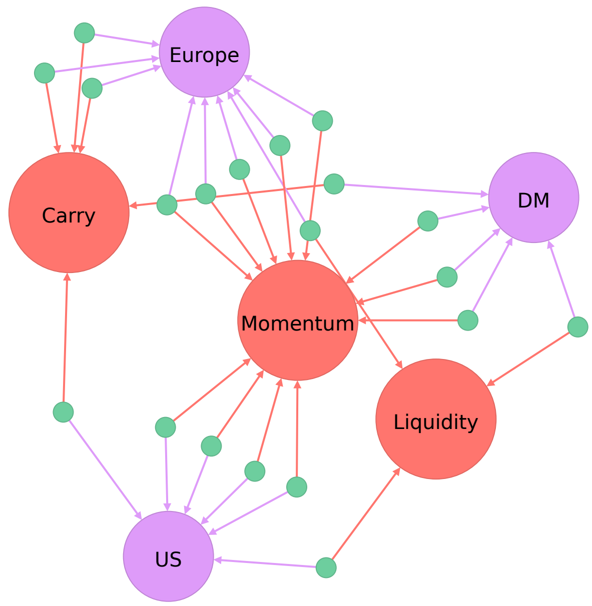


Figure 44: Graph database of commodity risk factors

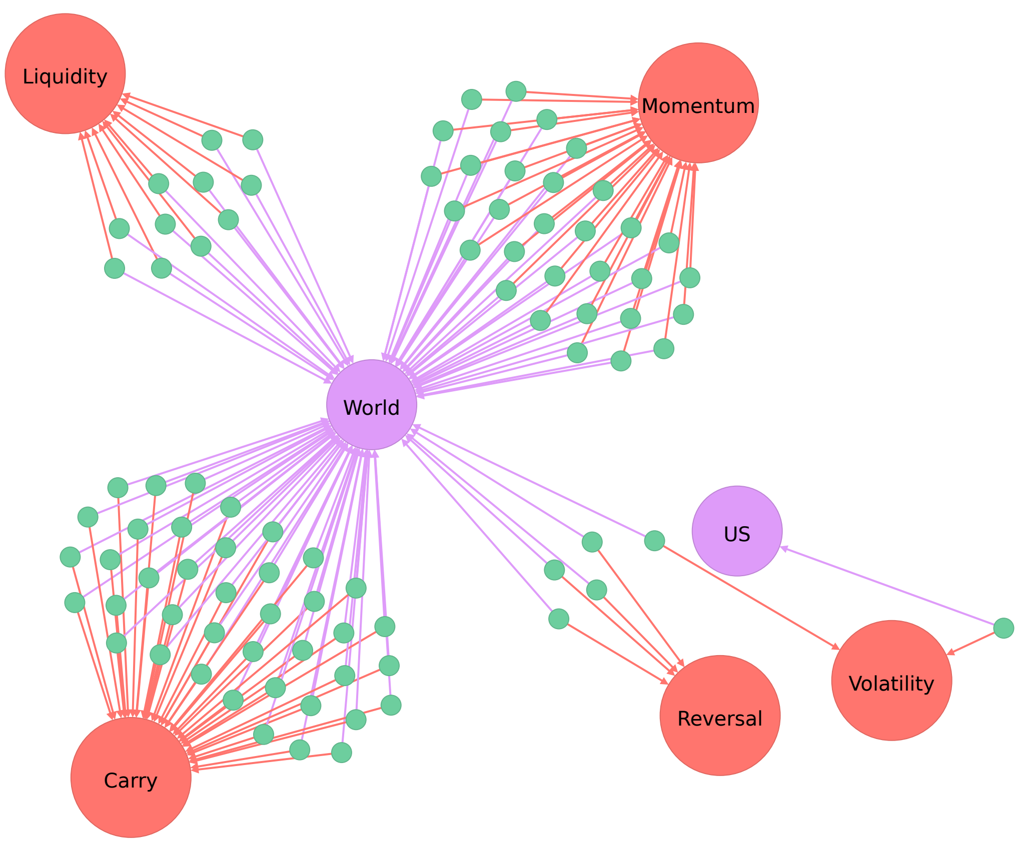
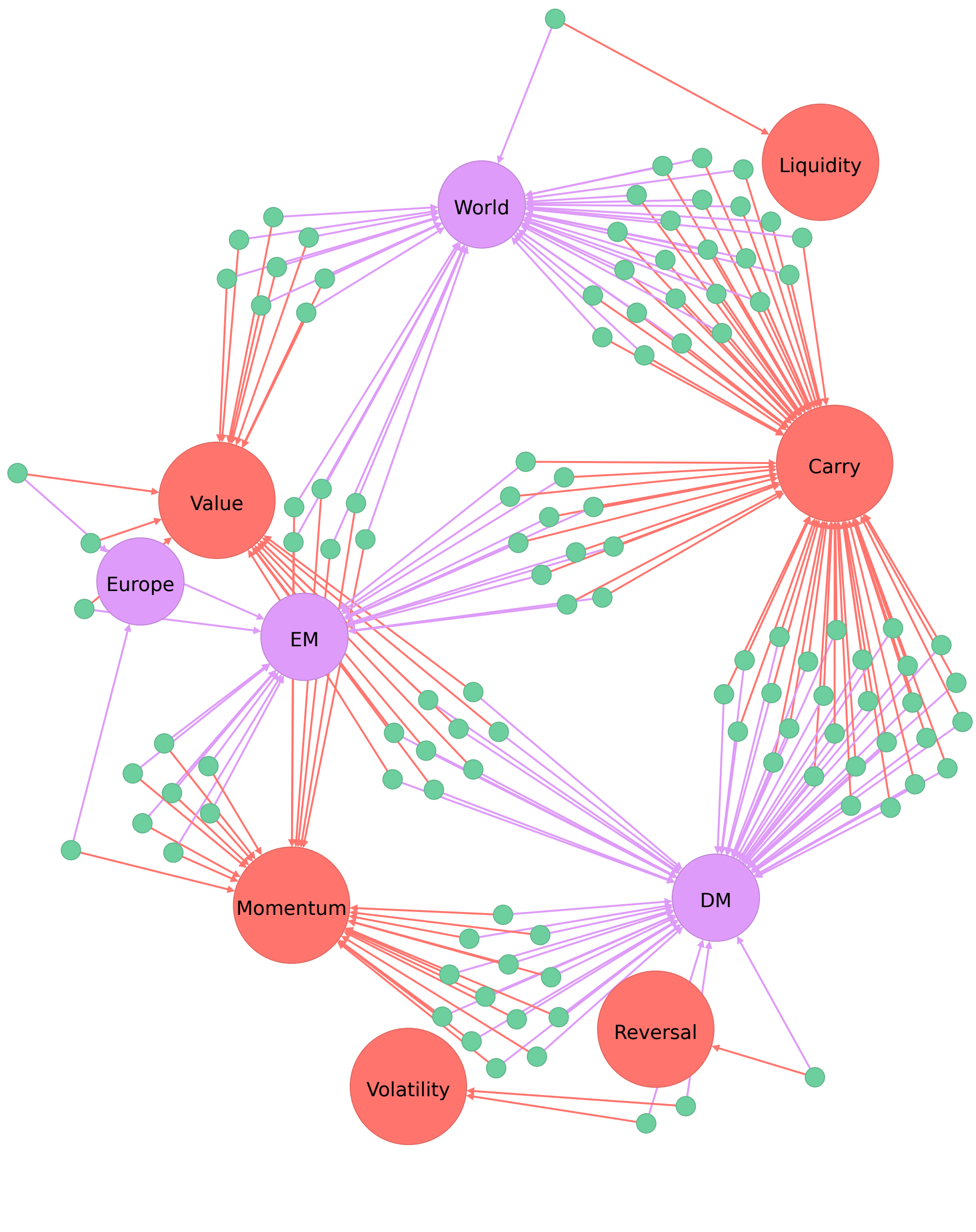


Figure 45: Graph database of currency risk factors



B Estimating the generic performance of alternative risk premia

One of the difficulties with alternative risk premia is building a track record or backtest that represents a good approximation of the true performance if one invests in these risk premia. Most academic backtests are not representative of what an investor can achieve in terms of returns, because they are not investable and are biased towards illiquid assets (small caps in the case of equity, low turnover for bonds, etc.). One famous example is the European HML risk factor calculated by Fama and French (2012). On average, the HML factor is calculated using a universe of more than 4 700 European stocks. Any seasoned investor knows that rebalancing such a portfolio is a very difficult task. In the case of the HML factor, the selection is performed once a year and we can expect that trading impacts are limited. However, when we use the same universe to build the momentum risk factor and a monthly rebalancing scheme, the resulting backtest has little significance and relevance for the investor, because of trading friction, rebalancing costs and investment capacity. Another issue concerns long/short risk premia. Systematic long/short portfolios are very popular in academic research and most alternative risk premia are analyzed in a long/short format. Unfortunately, long/short portfolios are difficult to implement in terms of management and they are even more sensitive to trading costs. Using indices is a better solution, because these serve as underlyings of investable products. In this case, the difference between the performance of the index and the performance of the investment is only due to the management fees. However, we must be careful with some indices, which could be optimized to present good historical performance (Bailey *et al.*, 2016). This bias is particularly present in indices designed by investment banks.

Most of the times, backtesting bias can be detected by due diligence. This explains that while the database contains more than 2 000 products, only two thirds are relevant and can be viewed as candidates mimicking alternative risk premia. Nevertheless, when we analyze the set of relevant indices for a given risk premium, we may observe large differences between their behavioral patterns in terms of performance, volatility, correlations, etc. For each risk premium, the idea is then to obtain a generic index, which represents the average performance. This is why the due diligence process must be complemented by a second quantitative step, which eliminates indices whose behavior is not in line with the average behavior of the universe. This second step is given in Algorithm 1, which describes the selection procedure after the first step. Given an initial set $\mathcal{P}$ of indices that replicate a given risk premium, the underlying idea is to find the set $\mathcal{S} \subset \mathcal{P}$, whose elements present very similar patterns. To this end, we consider a deletion algorithm by excluding at each iteration the index that is furthest from the average behavior of the other indices. The similarity is calculated using the R-squared statistic $\mathbf{R}_k^2$ of the linear regression between the return of the index $R_{k,t}$ and the average return of the other indices $R_t^{(-k)}$. The algorithm stops when the similarity statistic $\mathbf{R}_k^2$ is larger than a threshold $\mathbf{R}_{\min}^2$ for all the indices that belong to the selection set $\mathcal{S}$. Finally, we estimate the generic performance of the risk premium by averaging the return of all the selected indices.

Remark 8 *The selection procedure may be viewed as a regularization procedure of the information matrix associated to the initial universe $\mathcal{P}$. Another solution consists therefore in using the L_1 regularization proposed by Bruder et al. (2013).*

Let us illustrate the selection procedure with the example of the equities/volatility/carry/US risk premium. We only select long/short indices denominated in USD and whose type is excess return. We report the cumulative performance of the 14 indices in Figure 46.

Algorithm 1 Selection procedure

Require: $\mathcal{P}$ the universe of indices that replicate a given risk premium
Require: $n_{\min}$ the minimum number of indices that compose the generic risk premium
Require: $\mathbf{R}^2_{\min}$ the minimum value taken by the R-squared statistic

Initialize $\mathcal{S} \leftarrow \mathcal{P}$
Initialize $n_{\mathcal{S}} \leftarrow \operatorname{card} \mathcal{P}$
Initialize convergence $\leftarrow 0$
while not convergence **do**
  for any product $k \in \mathcal{S}$ **do**
    Perform the linear regression:

$$R_{k,t} = \alpha_k + \beta_k R_t^{(-k)} + \varepsilon_{k,t}$$

    where:

$$R_t^{(-k)} = \frac{1}{n_{\mathcal{S}} - 1} \sum_{\substack{i \in \mathcal{S} \\ i \neq k}} R_{i,t}$$

    Calculate the R-squared statistic:

$$\mathbf{R}^2_k = 1 - \frac{\sum_t \varepsilon^2_{k,t}}{\sum_t R^2_{k,t}}$$

  end for
  Find the index which has the minimum R-squared statistic:

$$k^\star = \arg \min_k \mathbf{R}^2_k$$

  if $n_{\mathcal{S}} = n_{\min}$ **or** $\mathbf{R}^2_{k^\star} \geq \mathbf{R}^2_{\min}$ **then**
    convergence $\leftarrow 1$
  else
    convergence $\leftarrow 0$
    $n_{\mathcal{S}} \leftarrow n_{\mathcal{S}} - 1$
    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{k^\star\}$
  end if
end while
return $\mathcal{S}$

We notice that the behavior is not homogenous across all indices. If we calculate the cross-correlation of daily returns, the minimum and maximum values are equal to -34.9% and 98.6%. The mean value is equal to 43.0% while the interquartile range is greater than 35%. By using the selection algorithm, we reduce the number of relevant indices. For instance, if the threshold $\mathbf{R}^2_{\min}$ is set to 30%, the number of selected indices is 10, if the threshold $\mathbf{R}^2_{\min}$ is set to 50%, the number of selected indices is seven, etc. Finally, we have five selected indices if we assume that $\mathbf{R}^2_{\min} = 70\%$. In Figure 47, we show the cumulative performance of the selection universe and the generic risk premium, which corresponds to the black line. In this case, the minimum and maximum values of the cross-correlation are equal to 90.0% and 98.6%. These five selected indices thus have high similarity and we can consider that their average behavior is a good proxy for what an investor may expect when he or she is exposed to the equities/volatility/carry/US risk premium.

Figure 46: Set $\mathcal{P}$ of the 14 equities/volatility/carry/US indices

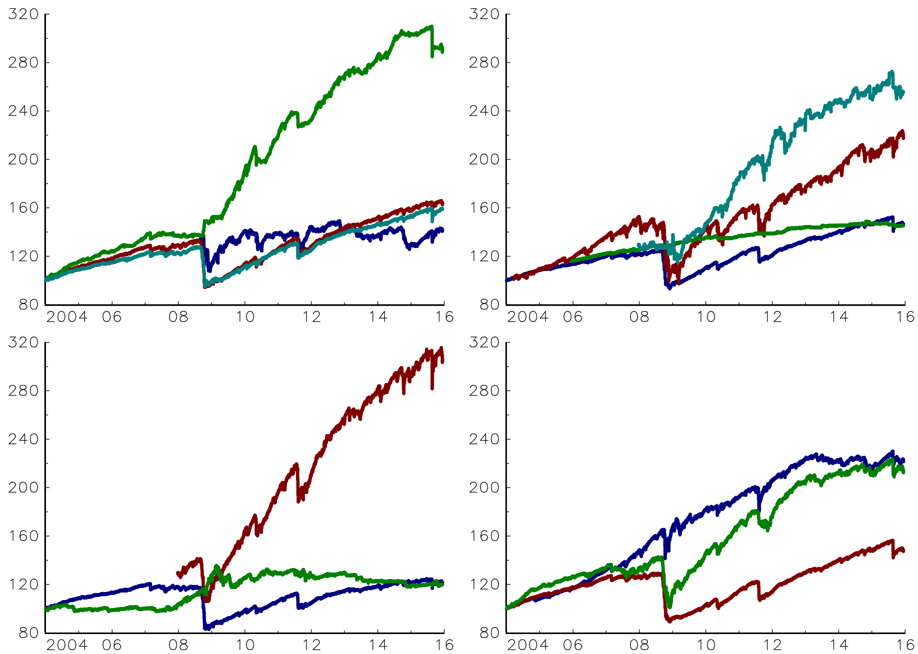
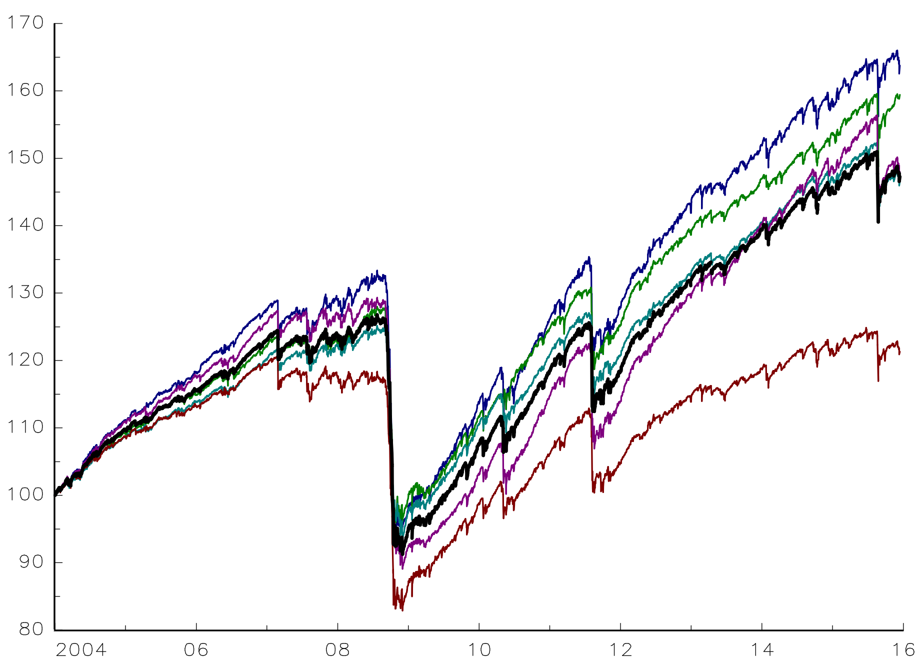


Figure 47: Selected indices and generic risk premium



C Mathematical results

C.1 The noncentral chi-squared distribution

C.1.1 Definition

Let $(X_1, \ldots, X_\nu)$ be a set of independent Gaussian random variables such that $X_i \sim \mathcal{N}\left(\mu_i, \sigma_i^2\right)$. The noncentral chi-squared random variable is defined as follows:

$$Y = \sum_{i=1}^{\nu} \frac{X_i^2}{\sigma_i^2}$$

We write $Y \sim \chi_\nu^2(\zeta)$ where ν is the number of degrees of freedom and ζ is the noncentrality parameter:

$$\zeta = \sum_{i=1}^{\nu} \frac{\mu_i^2}{\sigma_i^2}$$

When μ_i is equal to zero, Y becomes a (central) chi-squared distribution $\chi_\nu^2(0)$.

C.1.2 Some properties

The cumulative distribution function of Y is defined as:

$$\mathbf{F}(y; \nu, \zeta) = \Pr\{Y \leq y\} = \sum_{j=0}^{\infty} \frac{e^{-\zeta/2}\zeta^j}{2^j j!} \mathbf{F}(y; \nu + 2j, 0)$$

where $\mathbf{F}(y; \nu, 0)$ is the cumulative distribution function of the chi-squared distribution with ν degrees of freedom. We deduce that the probability density function is:

$$f(y; \nu, \zeta) = \sum_{j=0}^{\infty} \frac{e^{-\zeta/2}\zeta^j}{2^j j!} f(y; \nu + 2j, 0)$$

where $f(y; \nu, 0)$ is the probability density function of the chi-squared distribution. We may also show that the mean and the variance of Y are $\nu + \zeta$ and $2(\nu + 2\zeta)$, respectively. For the skewness and excess kurtosis coefficients, we obtain:

$$\begin{aligned}
\gamma_1 &= (\nu + 3\zeta)\sqrt{\frac{2^3}{(\nu + 2\zeta)^3}} \\
\gamma_2 &= \frac{12(\nu + 4\zeta)}{(\nu + 2\zeta)^2}
\end{aligned}$$

C.2 Reversal strategy with a price target

C.2.1 Definition

Let S_t be the price of an asset. We assume that S_t follows a geometric Brownian motion:

$$\mathrm{d}S_t = \mu_t S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t$$

The investment strategy is described by the number of assets $f(S_t)$ held at time t. The value of the portfolio X_t satisfies the following relationship:

$$\mathrm{d}X_t = f(S_t)\,\mathrm{d}S_t$$

Bruder and Gaussel (2011) show that the P&L of the strategy may be broken down into two components:

$$X_T - X_0 = \underbrace{F(S_t) - F(S_0)}_{\text{option profile}} + \underbrace{-\frac{1}{2}\int_0^T \sigma_t^2 f'(S_t) S_t^2 \,\mathrm{d}t}_{\text{trading impact}} \tag{17}$$

where $F(S) = \int_a^S f(s)\,\mathrm{d}s$. Let us consider the reversal strategy defined by:

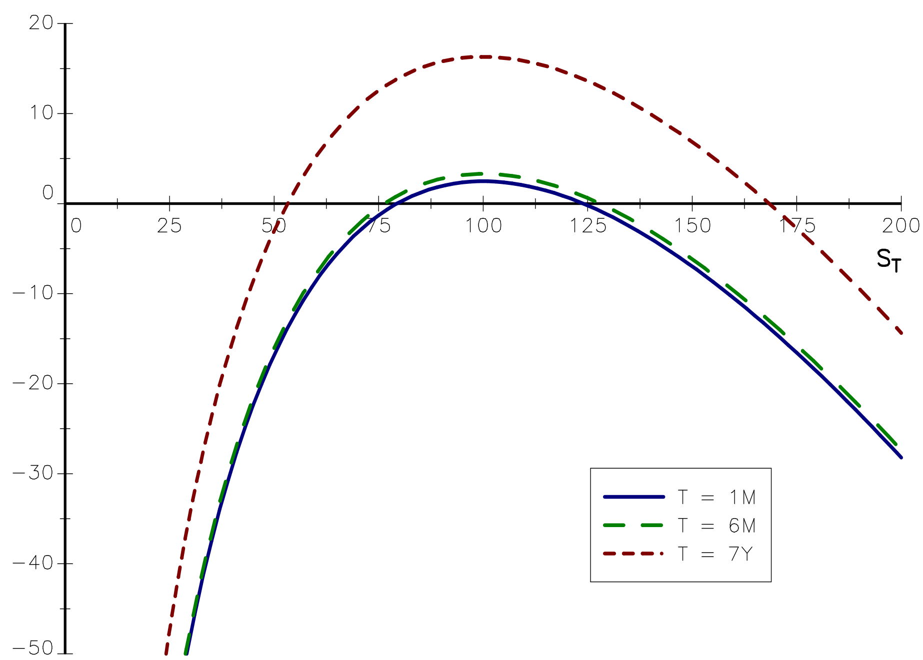
$$f(S_t) = m\frac{(\bar{S} - S_t)}{S_t}$$

where $\bar{S}$ is the price target of the asset and $m > 0$. If the current price is lower than the target level ($S_t \leq \bar{S}$), the nominal exposure $f(S_t) S_t$ is positive. On the contrary, we obtain a short exposure if the current price is higher than the target level. Using Equation (17), we obtain:

$$X_T - X_0 = m\bar{S}\ln\frac{S_T}{S_0} - m(S_T - S_0) + \frac{m}{2}\bar{S}\int_0^T \sigma_t^2 \,\mathrm{d}t$$

In Figure 48, we represent the expected gain of the strategy when the current price S_0 is 80, the target price $\bar{S}$ is 100, the volatility σ_t is 20% and the position size m is 1. We obtain a convex payoff with positive vega and theta. Therefore, the strategy benefits from the volatility risk.

Figure 48: Payoff of the reversal strategy



C.2.2 Statistical properties

Lemma 1 *In order to find the different moments, we use the following results*[60]*:*

$$\begin{aligned}
\mathbb{E}\left[e^{aWt}\right] &= e^{\frac{1}{2}a^2 t} \\
\mathbb{E}\left[e^{aW_t} W_t\right] &= ate^{\frac{1}{2}a^2 t} \\
\mathbb{E}\left[e^{aW_t} W_t^2\right] &= (a^2t^2 + t)e^{\frac{1}{2}a^2 t}
\end{aligned}$$

with $a > 0$.

Using the solution of S_T:

$$S_T = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right)$$

we obtain:

$$\begin{aligned}
X_T - X_0 &= m\bar{S}\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right) - m\left(S_T - S_0\right) + \frac{1}{2}m\bar{S}\sigma^2 T \\
&= m\left(-\left(S_T - S_0\right) + \mu\bar{S}T + \bar{S}\sigma W_T\right)
\end{aligned}$$

The expected return is equal to:

$$\mathbb{E}\left[X_T - X_0\right] = m\left(\mu\bar{S}T - S_0\left(e^{\mu T} - 1\right)\right)$$

We deduce that the centered process of $X_T - X_0$ is:

$$\left(X_T - X_0\right) - \mathbb{E}\left[X_T - X_0\right] = m\left(-\left(S_T - S_0 e^{\mu T}\right) + \bar{S}\sigma W_T\right)$$

The variance is defined as follows:

$$\operatorname{var}\left(X_T - X_0\right) = m^2\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)^2\right] - 2m^2\bar{S}\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)\sigma W_T\right] + m^2\bar{S}^2\sigma^2 T$$

Using the expression of S_T, we have:

$$S_T - S_0e^{\mu T} = S_0e^{\mu T}\left(e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - 1\right)$$

It follows that:

$$\begin{aligned}
\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)^2\right] &= S_0^2e^{2\mu T}\left(e^{\sigma^2 T} - 1\right) \\
\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)\sigma W_T\right] &= S_0e^{\mu T}\sigma^2 T
\end{aligned}$$

We finally obtain that:

$$\operatorname{var}\left(X_T - X_0\right) = m^2S_0^2e^{2\mu T}\left(e^{\sigma^2 T} - 1\right) - 2m^2S_0e^{\mu T}\bar{S}\sigma^2 T + m^2\bar{S}^2\sigma^2 T$$

The centered third moment is equal to:

$$\begin{aligned}
\mu_3\left(X_T - X_0\right) = &-m^3\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)^3\right] + 3m^3\bar{S}\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)^2\sigma W_T\right] - \\
&3m^3\bar{S}^2\mathbb{E}\left[\left(S_T - S_0e^{\mu T}\right)\sigma^2 W_T^2\right]
\end{aligned}$$

[60] The first formula is the classical Laplace transform of the Brownian motion while the other formulas can be derived from the expansion of e^{aW_t} as an infinite sum.

It follows that:

$$\begin{aligned}
\mathbb{E}\left[\left(S_T - S_0 e^{\mu T}\right)^3\right] &= S_0^3 e^{3\mu T}\left(e^{3\sigma^2 T} - 3e^{\sigma^2 T} + 2\right) \\
\mathbb{E}\left[\left(S_T - S_0 e^{\mu T}\right)^2 \sigma W_T\right] &= 2S_0^2 e^{2\mu T}\sigma^2 T\left(e^{\sigma^2 T} - 1\right) \\
\mathbb{E}\left[\left(S_T - S_0 e^{\mu T}\right)\sigma^2 W_T^2\right] &= S_0 e^{\mu T}\sigma^4 T^2
\end{aligned}$$

The expression of the centered third moment is then:

$$\begin{aligned}
\mu_3\left(X_T - X_0\right) &= -m^3 S_0^3 e^{3\mu T}\left(e^{3\sigma^2 T} - 3e^{\sigma^2 T} + 2\right) + 6m^3 \bar{S} S_0^2 e^{2\mu T}\sigma^2 T\left(e^{\sigma^2 T} - 1\right) - \\
&\quad 3m^3 \bar{S}^2 S_0 e^{\mu T}\sigma^4 T^2
\end{aligned}$$

We conclude that the skewness is:

$$\gamma_1\left(X_T - X_0\right) = -\frac{\left(e^{3\sigma^2 T} - 3e^{\sigma^2 T} + 2\right)x^3 - 6\bar{S}\sigma^2 T\left(e^{\sigma^2 T} - 1\right)x^2 + 3\bar{S}^2\sigma^4 T^2 x}{\left(\left(e^{\sigma^2 T} - 1\right)x^2 - 2\bar{S}\sigma^2 T x + \bar{S}^2\sigma^2 T\right)^{3/2}}$$

where $x = S_0 e^{\mu T}$ is a positive scalar. We notice that the sign of $\gamma_1\left(X_T - X_0\right)$ is given by:

$$\operatorname{sgn}\gamma_1\left(X_T - X_0\right) = -\operatorname{sgn}\left(\left(e^{3z} - 3e^z + 2\right)x^2 - 6\bar{S}z\left(e^z - 1\right)x + 3\bar{S}^2 z^2\right)$$

with $z = \sigma^2 T > 0$. We verify that $g(z) = e^{3z} - 3e^z + 2 > 0$ because $g(0) = 0$ and $g'(z) > 0$. This implies that the sign of the second-order polynomial is always positive if the discriminant of the quadratic equation is negative. As the discriminant of the quadratic equation is negative[61], the sign of the second-order polynomial is always positive and the skewness is negative.

In Figure 49, we report the probability density of $X_T - X_0$ for different parameters μ when the asset volatility is equal to 30%, m is equal to 1, $S_0 = 80$, $\bar{S} = 100$ and $T = 1$. We observe that the distribution has negative skewness. The moments are given in Figure 50. We observe that the mean is not necessarily positive. It depends on the position between the current price S_0 and the target price $\bar{S}$ and is also sensitive to the expected return μ of the asset. The skewness reaches its minimum around $\mu = \bar{S} - S_0$.

C.2.3 Extension to the Ornstein-Uhlenbeck process

The previous framework highlights the asymmetry between positive and negative gains. However, the strategy does not necessarily have a positive expected value. Therefore, we consider a more realistic framework when the asset return satisfies the mean-reverting property.

We assume that S_t is an Ornstein-Uhlenbeck process given by the following stochastic differential equation:

$$\mathrm{d}S_t = \alpha\left(S_\infty - S_t\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

where S_∞ is the long-term mean of S_t and $\alpha > 0$ is the speed of mean reversion. The solution is:

$$S_t = S_0 e^{-\alpha t} + S_\infty\left(1 - e^{-\alpha t}\right) + \sigma\int_0^t e^{\alpha(u-t)}\,\mathrm{d}W_u$$

[61] We have $\Delta = 12\bar{S}^2 z^2 (1 - e^z)^3 < 0$.

Figure 49: Probability density function of $X_T - X_0$

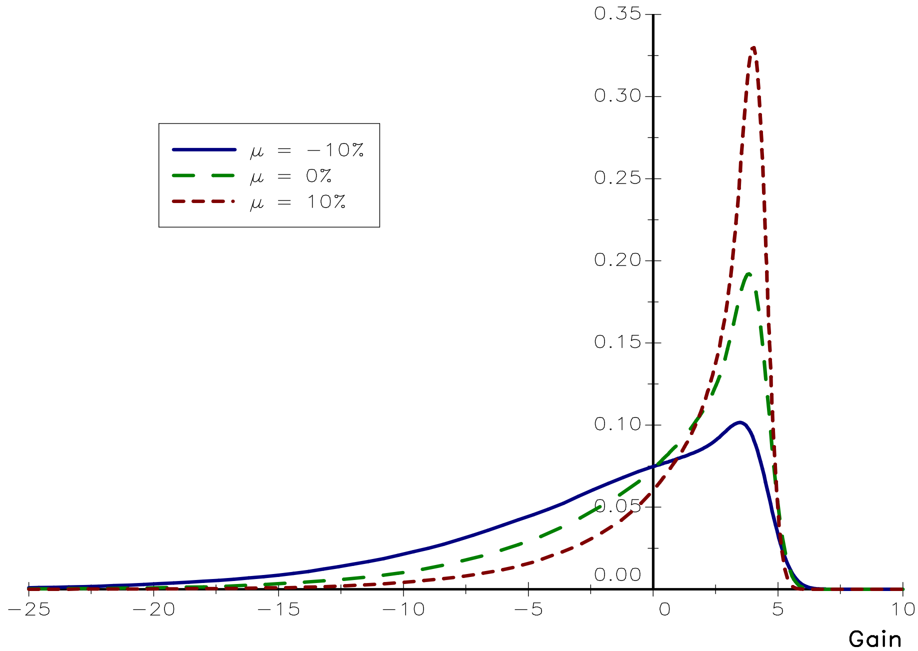
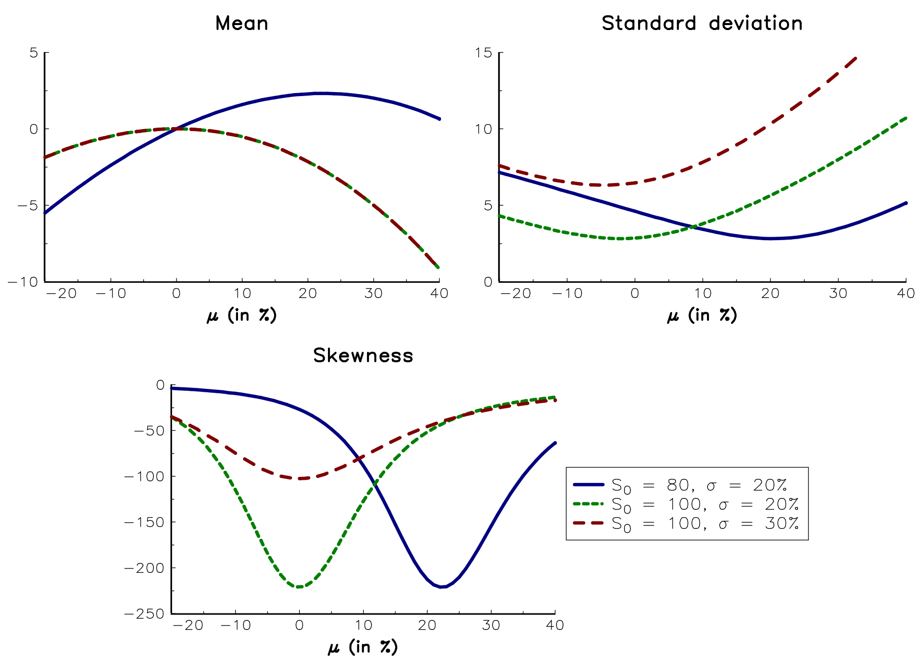


Figure 50: Moments of $X_T - X_0$ with respect to the parameter μ



We can show that it is a Gaussian process:

$$S_t = a_t + b_t X$$

with $a_t = S_0 e^{-\alpha t} + S_\infty \left(1 - e^{-\alpha t}\right)$, $b_t = \sigma \sqrt{1 - e^{-2\alpha t}} / \sqrt{2\alpha}$ and $X \sim \mathcal{N}(0, 1)$.

We consider the following reversal strategy:

$$f(S_t) = m\left(\bar{S} - S_t\right)$$

where $\bar{S}$ is the target value of the trading strategy, meaning that the portfolio is short the asset when $S_t > \bar{S}$ (and, conversely, long the asset when $S_t < \bar{S}$). It follows that the P&L of the strategy is:

$$X_T - X_0 = m\bar{S}\left(S_T - S_0\right) - \frac{m}{2}\left(S_T^2 - S_0^2\right) + \frac{m}{2}\sigma^2 T$$

In order to calculate the different moments, we use the following result:

$$\begin{aligned} \mathbb{E}\left[S_t^r\right] &= \mathbb{E}\left[\left(a_t + b_t X\right)^r\right] \\ &= \sum_{k=0}^{r} \binom{r}{k} a_t^{r-k} b_t^k \mathbb{E}^k\left[X\right] \end{aligned}$$

with:

$$\mathbb{E}^k\left[X\right] = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (k-1)!! & \text{if } r \text{ is even} \end{cases}$$

The expected value of the P&L We find that[62]:

$$\begin{aligned} \mathbb{E}\left[X_T - X_0\right] &= m\left(1 - e^{-\alpha T}\right)\left(S_\infty - S_0\right)\left(\left(\bar{S} - \frac{S_0 + S_\infty}{2}\right) + \frac{1}{2}e^{-\alpha T}\left(S_\infty - S_0\right)\right) + \\ & \quad \frac{m}{2}\left(1 - \frac{1}{2\alpha T} + \frac{e^{-2\alpha t}}{2\alpha T}\right)\sigma^2 T \end{aligned}$$

This expression highlights the relationship between the expected P&L and the parameters $\bar{S}$ and S_∞. If we assume that T is large, we obtain:

$$\lim_{T \to \infty} \mathbb{E}\left[X_T - X_0\right] = m\left(S_\infty - S_0\right)\left(\bar{S} - \frac{S_0 + S_\infty}{2}\right) + \frac{m}{2}\sigma^2 T$$

The first term is positive when $\bar{S}$ is closer to S_∞ than to S_0. This is consistent with the intuition that the strategy works better when $\bar{S}$ is an estimate of the long-term mean S_∞. We also notice that the expected P&L does not reach its maximum when $\bar{S} = S_\infty$. Indeed, if $S_\infty > S_0$, it is an increasing function of $\bar{S}$. This is due to the profile of $f(S_t)$:

- If $S_\infty > S_0$, the expected trend of S_t is positive and the investors make a profit when $\bar{S} > S_0$.
- This profit depends on the magnitude of the exposure $\bar{S} - S_t$. The higher the leverage, the higher the profit.

[62] We use the results $\mathbb{E}\left[S_T\right] = a_T$ and $\mathbb{E}\left[S_T^2\right] = a_T^2 + b_T^2$.

The skewness of the P&L It follows that[63]:

$$\operatorname{var}\left(X_T - X_0\right) = \left(\bar{S} - a_T\right)^2 b_T^2 + \frac{1}{2} b_T^4$$

and:

$$\mu_3\left(X_T - X_0\right) = -3\left(\bar{S} - a_T\right)^2 b_T^4 - b_T^6$$

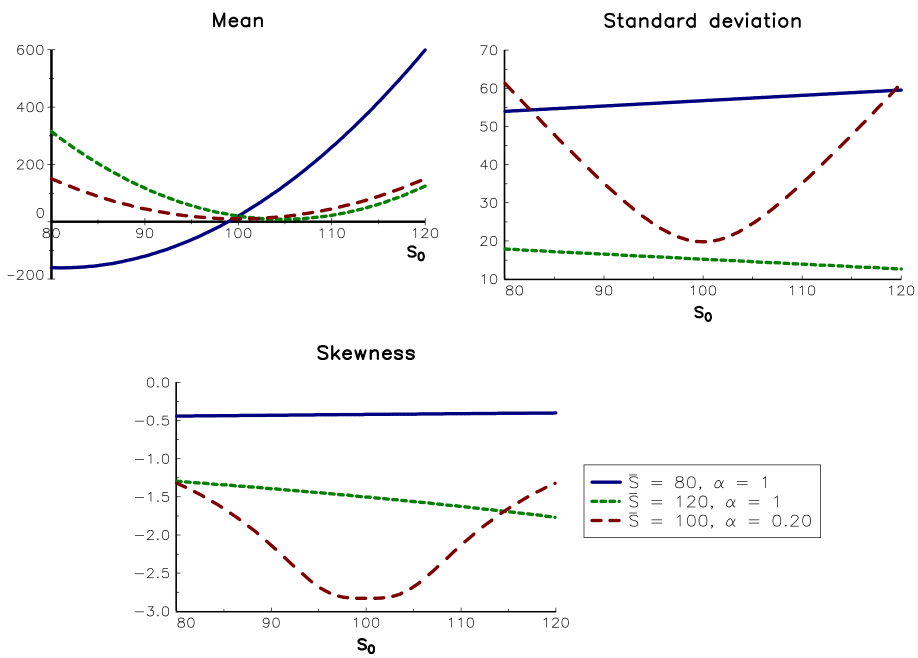
We finally obtain the expression of the skewness:

$$\gamma_1\left(X_T - X_0\right) = -\frac{3\left(\bar{S} - a_T\right)^2 b_T^4 + b_T^6}{\left(\left(\bar{S} - a_T\right)^2 b_T^2 + \frac{1}{2} b_T^4\right)^{3/2}}$$

Because $b_T \geq 0$, we deduce that the skewness of the strategy is negative.

In Figure 51, we report the moments of $X_T - X_0$ with respect to the current value S_0 when the asset volatility is equal to 4, m is equal to 1 and $T = 3$. We observe that the mean is not necessarily positive. It depends on the position between the current price S_0, the target price $\bar{S}$ and the long-term price S_∞. We also notice the impact of the mean-reversion parameter α. We verify that the skewness is negative.

Figure 51: Moments of $X_T - X_0$ with respect to the current price S_0



[63] We have $\mathbb{E}\left[S_T^3\right] = a_T^3 + 3a_T b_T^2$, $\mathbb{E}\left[S_T^4\right] = a_T^4 + 6a_T^2 b_T^2 + 3b_T^4$, $\mathbb{E}\left[S_T^5\right] = a_T^5 + 10a_T^3 b_T^2 + 15a_T b_T^4$ and $\mathbb{E}\left[S_T^6\right] = a_T^6 + 15a_T^4 b_T^2 + 45a_T^2 b_T^4 + 15b_T^6$.

C.3 Trend-following strategy with an EWMA trend

C.3.1 Definition

We assume that S_t follows a geometric Brownian motion with constant volatility, but a time-varying unobservable trend:

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t \\ \mathrm{d}\mu_t = \gamma \,\mathrm{d}W_t^\star \end{cases}$$

We estimate the trend using the exponentially moving average estimator defined as follows:

$$\hat{\mu}_t = \lambda \int_0^t e^{-\lambda(t-s)} \,\mathrm{d}y_s + e^{-\lambda t}\hat{\mu}_0$$

where $y_t = \ln S_t$ and $\lambda = \gamma/\sigma$. The trend-following strategy is defined by the following nominal exposure:

$$\frac{\mathrm{d}X_t}{X_t} = m\hat{\mu}_t \frac{\mathrm{d}S_t}{S_t}$$

where m is the parameter of position sizing. The exposure is an increasing function of the estimated trend. In particular, we obtain a long portfolio if $\hat{\mu}_t > 0$ and a short portfolio otherwise. Bruder and Gaussel (2011) show that the performance of the trend-following strategy at time t is equal to:

$$\ln \frac{X_T}{X_0} = m\frac{\left(\hat{\mu}_T^2 - \hat{\mu}_0^2\right)}{2\lambda} + \frac{m}{2}\left(\int_0^T \hat{\mu}_t^2 \left(2 - m\sigma^2\right) \,\mathrm{d}t - \lambda\sigma^2 T\right)$$

In Figure 52, we report the annualized payoff of the strategy with respect to the realized trend $\hat{\mu}_T$ when the current trend is 30%, the volatility is 20%, the maturity is one year and the position size m is equal to 1. We observe that we obtain a concave profile[64]. The strategy has a negative vega and this volatility risk is a decreasing function of the moving average duration $\tau = \lambda^{-1}$.

C.3.2 Statistical properties

Let us determine the probability distribution of the annualized return of G_t:

$$G_t = \frac{m}{2}\left(\hat{\mu}_t^2 \left(2 - m\sigma^2\right) - \lambda\sigma^2\right)$$

We note $\hat{s}_t = \hat{\mu}_t/\sigma$ the estimator of the instantaneous Sharpe ratio. We notice that:

$$\hat{s}_t \sim \mathcal{N}\left(s_t, \frac{\lambda}{2}\right)$$

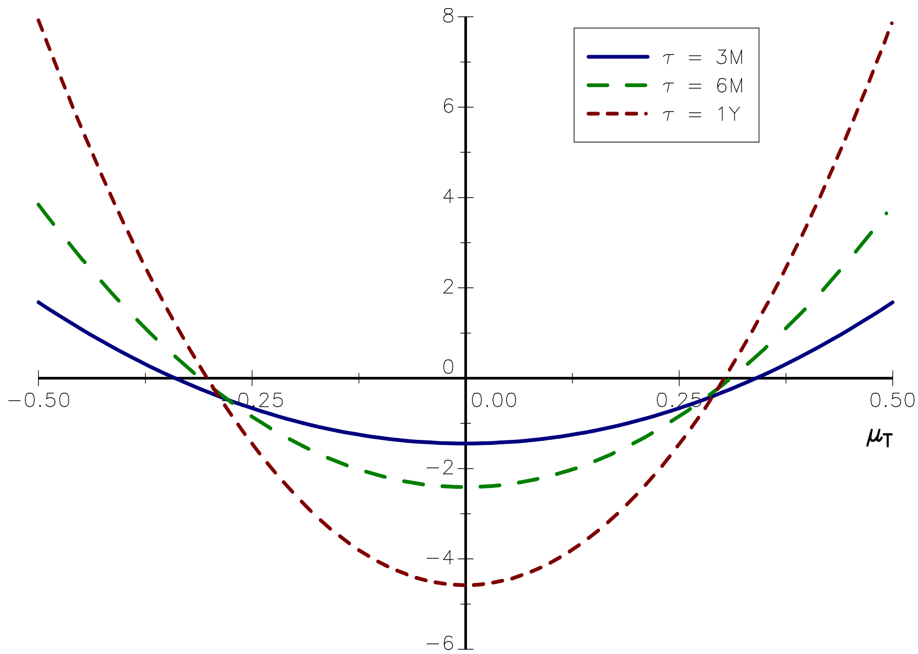
with $s_t = \mu_t/\sigma$. It follows that $\hat{s}_t^2$ is a noncentral chi-squared random variable $\chi_1^2(\zeta)$ with:

$$\zeta = 2\frac{s_t^2}{\lambda}$$

[64] We use the fact that the unconditional mathematical expectation of $\hat{\mu}_t^2$ is:

$$\mathbb{E}\left[\hat{\mu}_t^2\right] = \mathbb{E}\left[\left(\lambda \int_{-\infty}^t e^{-\lambda(t-s)} \,\mathrm{d}y_s\right)^2\right] = \frac{\lambda\sigma^2}{2}$$

Figure 52: Payoff (in %) of the trend-following strategy



We can write G_t as follows:

$$G_t = \frac{m\sigma^2}{2}\left(2 - m\sigma^2\right)\hat{s}_t^2 - \frac{m}{2}\lambda\sigma^2$$

We deduce that G_t is an affine transformation of the noncentral chi-squared distribution[65]:

$$\Pr\left\{G_t \leq g\right\} = \mathbf{F}\left(\frac{4g + 2m\lambda\sigma^2}{m\lambda\sigma^2\left(2 - m\sigma^2\right)}; 1, \zeta\right)$$

Concerning the moments, we have:

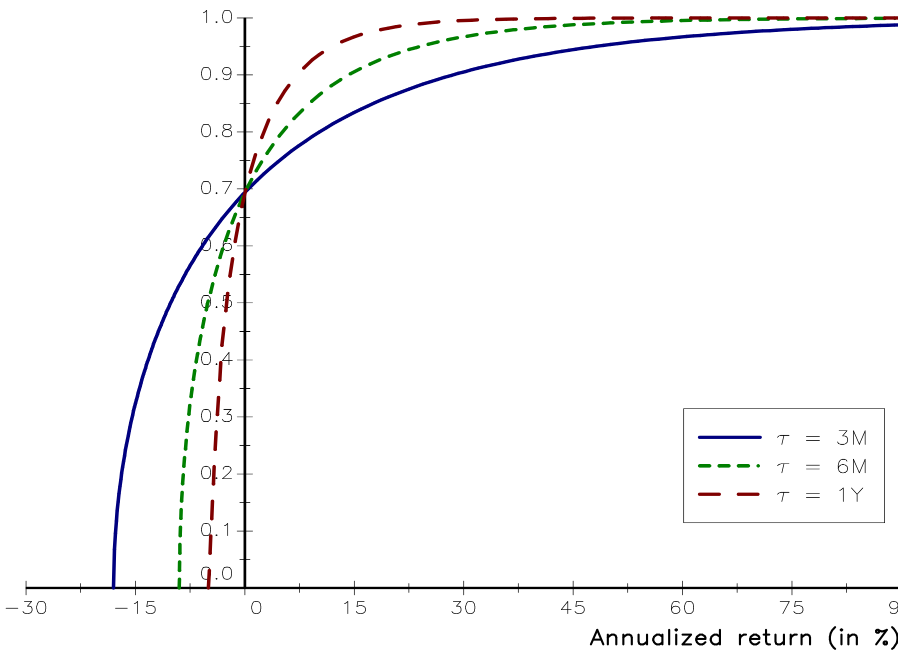
$$\begin{array}{rcl} \mu\left(G_t\right) & = & \dfrac{m\sigma^2}{2}\left(2 - m\sigma^2\right)\left(1 + \zeta\right) - \dfrac{m\lambda\sigma^2}{2} \\ \sigma\left(G_t\right) & = & \left|\dfrac{m\sigma^2}{2}\left(2 - m\sigma^2\right)\right|\sqrt{2 + 4\zeta} \\ \gamma_1\left(G_t\right) & = & \left(6 + 6\zeta\right)\sqrt{\dfrac{2}{\left(1 + 2\zeta\right)^3}} \\ \gamma_2\left(G_t\right) & = & \dfrac{12 + 48\zeta}{\left(1 + 2\zeta\right)^2} \end{array}$$

In Figures 53 and 54, we report the cumulative distribution function of G_t for different decay parameters τ when the asset volatility is equal to 30% and m is equal to 1. We consider two cases: the expected return μ_t is equal to zero in Figure 53 whereas it is equal

[65] We assume that m satisfies the inequalities $0 < m < {}^2/_{\sigma^2}$. Otherwise, it does not correspond to a trend-following strategy ($m < 0$) or the gamma costs due to position sizing are too large ($m > {}^2/_{\sigma^2}$).

to 60% in Figure 54. We observe that the distribution has a positive skewness, especially when the performance of the asset is poor. This is confirmed by the plot of the moments (Figure 55). The moments are symmetrical with respect to the zero Sharpe ratio. The important quantity is therefore the absolute value of the expected return. The frequency of the EWMA trend has little influence on the mean, but a large impact on the three other moments. The skewness and excess kurtosis statistics are maximum when the asset has no trend.

Figure 53: Cumulative distribution function of G_t ($s_t = 0$)



C.4 Expression of the carry in the case of fixed income

We recall that the price of a zero-coupon bond with maturity date T is equal to $B_t(T) = e^{-(T-t)R_t(T)}$ where $R_t(T)$ is the corresponding zero-coupon rate. We deduce that the duration $D_t(T)$ is equal to:

$$\begin{aligned} D_t(T) &= -\frac{1}{B_t(T)} \frac{\partial\, B_t(T)}{\partial\, R_t(T)} \\ &= (T-t) \end{aligned}$$

For the convexity $C_t(T)$, we obtain:

$$\begin{aligned} C_t(T) &= \frac{1}{B_t} \frac{\partial^2\, B_t(T)}{\partial\, R_t^2(T)} \\ &= (T-t)^2 \end{aligned}$$

Figure 54: Cumulative distribution function of G_t ($s_t = 2$)

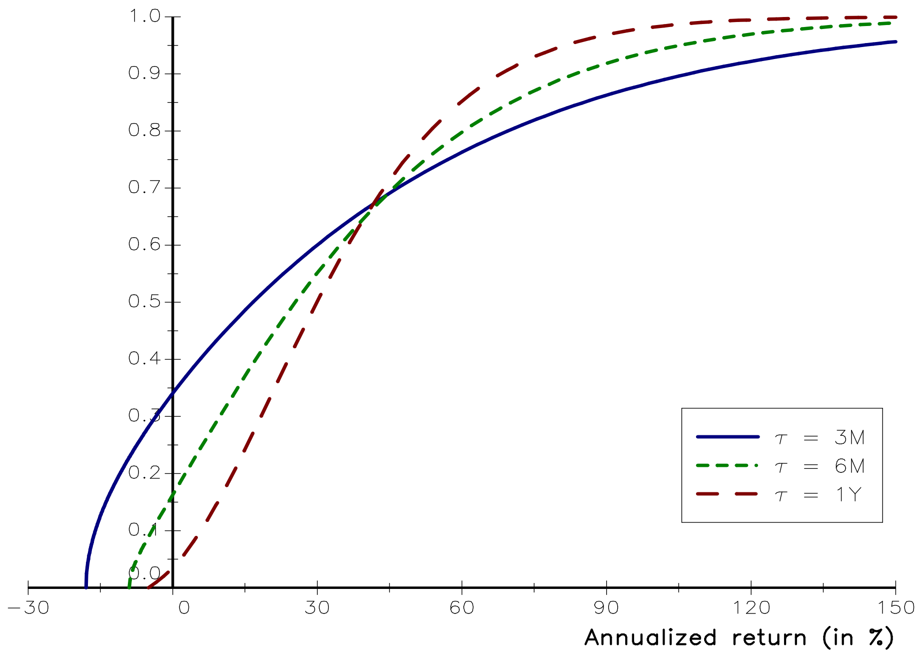
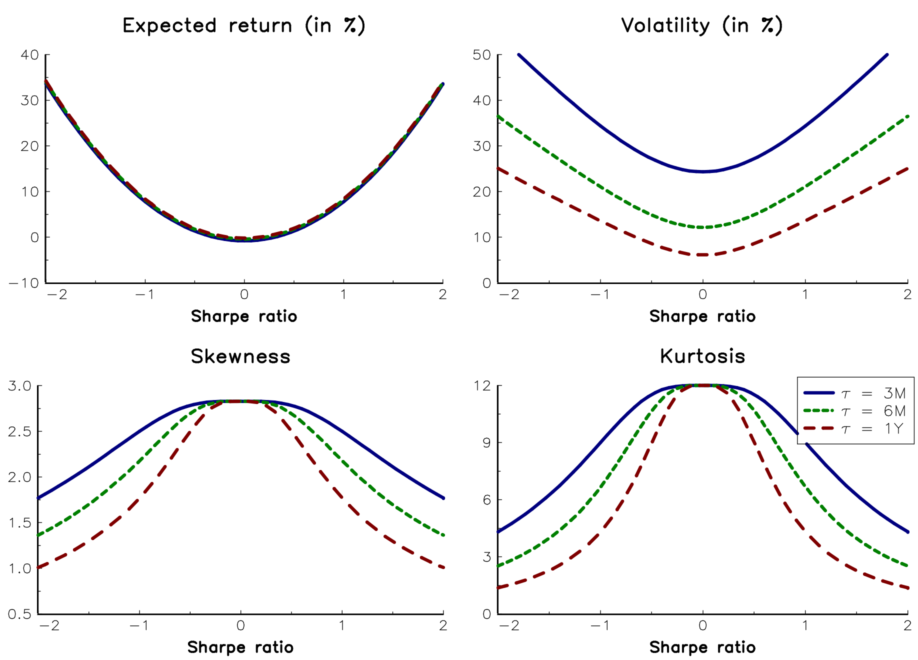


Figure 55: Moments of G_t with respect to the Sharpe ratio s_t



C.4.1 Bond carry

Let us apply Ito's lemma to the price $B_t(T)$:

$$\frac{\mathrm{d}B_t(T)}{B_t(T)} = -D_t(T)\,\mathrm{d}R_t(T) + R_t(T)\,\mathrm{d}t + \frac{1}{2}C_t(T)\,\langle \mathrm{d}R_t(T)\rangle$$

If we buy a zero-coupon bond with a zero-coupon rate $R_t(T)$ and sell it one day later, the new zero-coupon rate is $R_{t+\mathrm{d}t}(T)$. In order to take this roll down effect into account, we replace the variation of the zero-coupon rate $R_t(T)$ with the variation due to the maturity decrease and the variation due to the curve:

$$\underbrace{\mathrm{d}\bar{R}_t(\bar{T})}_{\text{variation of the CM rate}} = \underbrace{\mathrm{d}R_t(T)}_{\text{variation of the ZC rate}} + \underbrace{\partial_{\bar{T}}\,\bar{R}_t(\bar{T})\,\mathrm{d}t}_{\text{instantaneous slope}}$$

where $\bar{R}_t(\bar{T})$ is the zero-coupon rate with a constant time to maturity $\bar{T} = T - t$. Using the relationship between the bond futures contract and the bond:

$$\frac{\mathrm{d}F_t(T)}{F_t(T)} = \frac{\mathrm{d}B_t(T)}{B_t(T)} - r_t\,\mathrm{d}t$$

we deduce that:

$$\frac{\mathrm{d}F_t(T)}{F_t(T)} = \underbrace{-\,D_t(T)\,\mathrm{d}\bar{R}_t(\bar{T})}_{\text{shift}} + \underbrace{\left(\bar{R}_t(\bar{T}) - r_t\right)\mathrm{d}t}_{\text{term premium}} + \underbrace{\partial_{\bar{T}}\,\bar{R}_t(\bar{T})\,\mathrm{d}t}_{\text{roll down}} + \underbrace{\frac{1}{2}C_t(T)\left\langle \mathrm{d}\bar{R}_t(\bar{T})\right\rangle}_{\text{convexity}}$$

Under the hypothesis $(\mathcal{H})$ that the yield curve does not move, we have $\mathrm{d}\bar{R}_t(\bar{T}) = 0$ meaning that the instantaneous carry of the bond futures contract is defined as follows:

$$\mathcal{C}_t = \frac{1}{\mathrm{d}t}\cdot\left.\frac{\mathrm{d}F_t(T)}{F_t(T)}\right|\mathrm{d}\bar{R}_t(\bar{T}) = 0$$

We finally obtain[66]:

$$\mathcal{C}_t = \underbrace{R_t(T) - r_t}_{\text{term premium}} + \underbrace{\partial_{\bar{T}}\,\bar{R}_t(\bar{T})}_{\text{roll down}} \tag{18}$$

C.4.2 Carry of the slope

We consider a strategy that is long on the bond with the longest maturity T_2 and short on the bond with the shortest maturity T_1. By using bond futures contracts, the return of this strategy is:

$$\frac{\mathrm{d}V_t}{V_t} = \frac{\mathrm{d}B_t(T_2)}{B_t(T_1)} - w_t\frac{\mathrm{d}B_t(T_1)}{B_t(T_1)} - (1-w_t)\,r_t\,\mathrm{d}t$$

where w_t is the relative weight of the short leg with respect to the long leg. One generally considers a duration hedging investment, meaning that:

$$w_t = \frac{D_t(T_2)}{D_t(T_1)}$$

[66]By construction, we have $\bar{R}_t(\bar{T}) = R_t(T)$.

Using the results obtained in the previous paragraph, we have:

$$\begin{aligned}
\frac{\mathrm{d}V_t}{V_t} &= -D_t\left(T_2\right)\mathrm{d}\bar{R}_t\left(\bar{T}_2\right)+\bar{R}_t\left(\bar{T}_2\right)\,\mathrm{d}t+\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_2\right)\,\mathrm{d}t+\frac{1}{2}\mathcal{C}_t\left(T_2\right)\left\langle \mathrm{d}\bar{R}_t\left(\bar{T}_2\right)\right\rangle - \\
&\quad \frac{D_t\left(T_2\right)}{D_t\left(T_1\right)}\left(-D_t\left(T_1\right)\mathrm{d}\bar{R}_t\left(\bar{T}_1\right)+\bar{R}_t\left(\bar{T}_1\right)\,\mathrm{d}t+\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_1\right)\,\mathrm{d}t+\frac{1}{2}\mathcal{C}_t\left(T_1\right)\left\langle \mathrm{d}\bar{R}_t\left(\bar{T}_1\right)\right\rangle\right) - \\
&\quad \left(1-\frac{D_t\left(T_2\right)}{D_t\left(T_1\right)}\right)r_t\,\mathrm{d}t \\
&= D_t\left(T_2\right)\left(-\left(\mathrm{d}\bar{R}_t\left(\bar{T}_2\right)-\mathrm{d}\bar{R}_t\left(\bar{T}_1\right)\right)+\frac{1}{2}\left(D_t\left(T_2\right)\left\langle \mathrm{d}\bar{R}_t\left(\bar{T}_2\right)\right\rangle - D_t\left(T_1\right)\left\langle \mathrm{d}\bar{R}_t\left(\bar{T}_1\right)\right\rangle\right)\right)+ \\
&\quad D_t\left(T_2\right)\left(\left(\frac{\bar{R}_t\left(\bar{T}_2\right)-r_t}{D_t\left(T_2\right)}-\frac{\bar{R}_t\left(\bar{T}_1\right)-r_t}{D_t\left(T_1\right)}\right)+\left(\frac{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_2\right)}{D_t\left(T_2\right)}-\frac{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_1\right)}{D_t\left(T_1\right)}\right)\right)\mathrm{d}t
\end{aligned}$$

It follows that:

$$\begin{aligned}
\frac{1}{D_t\left(T_2\right)}\cdot\frac{\mathrm{d}V_t}{V_t} &= \underbrace{-\left(\mathrm{d}\bar{R}_t\left(\bar{T}_2\right)-\mathrm{d}\bar{R}_t\left(\bar{T}_1\right)\right)}_{\text{slope variation}}+ \\
&\quad \underbrace{\frac{1}{2}\left(D_t\left(T_2\right)\left\langle \mathrm{d}\bar{R}_t\left(\bar{T}_2\right)\right\rangle - D_t\left(T_1\right)\left\langle \mathrm{d}\bar{R}_t\left(\bar{T}_1\right)\right\rangle\right)}_{\text{volatility difference}}+ \\
&\quad \underbrace{\left(\frac{\bar{R}_t\left(\bar{T}_2\right)-r_t}{D_t\left(T_2\right)}-\frac{\bar{R}_t\left(\bar{T}_1\right)-r_t}{D_t\left(T_1\right)}\right)\mathrm{d}t}_{\text{duration neutral slope}}+\underbrace{\left(\frac{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_2\right)}{D_t\left(T_2\right)}-\frac{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_1\right)}{D_t\left(T_1\right)}\right)\mathrm{d}t}_{\text{duration neutral roll down}}
\end{aligned}$$

Under the hypothesis $(\mathcal{H})$, we deduce that the instantaneous carry of the slope strategy is equal to:

$$\mathcal{C}_t=\underbrace{\left(R_t\left(T_2\right)-r_t\right)-\frac{D_2\left(T_1\right)}{D_t\left(T_1\right)}\left(R_t\left(T_1\right)-r_t\right)}_{\text{duration neutral slope}}+\underbrace{\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_2\right)-\frac{D_2\left(T_1\right)}{D_t\left(T_1\right)}\partial_{\bar{T}}\,\bar{R}_t\left(\bar{T}_1\right)}_{\text{duration neutral roll down}} \tag{19}$$

C.5 Time-aggregation of the AR(1) process

We assume that:

$$X_t=\rho X_{t-1}+\varepsilon_t$$

where $|\rho|<1$, $\varepsilon_t\sim\mathcal{N}\left(0,\sigma_\varepsilon^2\right)$ and $\operatorname{cov}\left(\varepsilon_t,\varepsilon_{t-j}\right)=0$ for $j\geq 1$. We have:

$$\begin{aligned}
X_t &= \rho X_{t-1}+\varepsilon_t \\
&= \rho^2 X_{t-2}+\rho\varepsilon_{t-1}+\varepsilon_t \\
&= \rho^3 X_{t-3}+\rho^2\varepsilon_{t-2}+\rho\varepsilon_{t-1}+\varepsilon_t
\end{aligned}$$

We deduce that:

$$X_t=\rho^h X_{t-h}+\sum_{j=0}^{h-1}\rho^j\varepsilon_{t-j}$$

The Wold decomposition of the process $X_t$is then:

$$X_t=\sum_{j=0}^{\infty}\rho^j\varepsilon_{t-j}$$

C.5.1 Autocovariance function

The variance $\gamma(0)$ of the AR(1) process is equal to:

$$\begin{aligned}\gamma(0) &= \operatorname{var}\left(\sum\nolimits_{j=0}^{\infty} \rho^j \varepsilon_{t-j}\right) \\ &= \sum\nolimits_{j=0}^{\infty} \rho^{2j} \operatorname{var}(\varepsilon_{t-j}) \\ &= \frac{\sigma_\varepsilon^2}{1-\rho^2}\end{aligned}$$

Let $\gamma(h)$ be the covariance between X_t and X_{t-h}. We have:

$$\begin{aligned}\gamma(h) &= \mathbb{E}\left[X_t X_{t-h}\right] \\ &= \mathbb{E}\left[\left(\rho^h X_{t-h} + \sum\nolimits_{j=0}^{h-1} \rho^j \varepsilon_{t-j}\right) X_{t-h}\right] \\ &= \rho^h \gamma(0)\end{aligned}$$

Because the process is stationary, we also have:

$$\operatorname{cov}(X_{t-h}, X_{t-k}) = \rho^{|h-k|} \gamma(0)$$

C.5.2 Variance of the simple moving average

We consider the simple moving average:

$$\bar{X}_t^{(h)} = \frac{1}{h} \sum_{j=0}^{h-1} X_{t-j}$$

where h is the order or the period length of the moving average. We have:

$$\begin{aligned}\operatorname{var}\left(\bar{X}_t^{(h)}\right) &= \frac{1}{h^2}\mathbb{E}\left[\sum\nolimits_{j=0}^{h-1} X_{t-j} \sum\nolimits_{j=0}^{h-1} X_{t-j}\right] \\ &= \frac{1}{h^2}\mathbb{E}\left[\sum\nolimits_{j=0}^{h-1} \sum\nolimits_{k=0}^{h-1} X_{t-j} X_{t-k}\right] \\ &= \frac{1}{h^2} \sum\nolimits_{j=0}^{h-1} \sum\nolimits_{k=0}^{h-1} \mathbb{E}\left[X_{t-j} X_{t-k}\right] \\ &= \frac{\gamma(0)}{h^2} \sum\nolimits_{j=0}^{h-1} \sum\nolimits_{k=0}^{h-1} \rho^{|h-k|}\end{aligned}$$

We finally deduce that[67]:

$$\begin{aligned}\operatorname{var}\left(\bar{X}_t^{(h)}\right) &= \frac{\gamma(0)}{h^2}\left(h + 2\sum\nolimits_{j=1}^{h-1} (h-j)\rho^j\right) \\ &= \frac{\gamma(0)}{h}\left(1 + 2\rho\frac{1-\rho^{h-1}}{1-\rho} - 2\sum\nolimits_{j=1}^{h-1} \frac{j}{h}\rho^j\right)\end{aligned}$$

[67] We use the following result:

$$\begin{aligned}\sum\nolimits_{j=1}^{h-1} (h-j)\rho^j &= h\sum\nolimits_{j=1}^{h-1} \rho^j - \sum\nolimits_{j=1}^{h-1} j\rho^j \\ &= h\frac{\rho-\rho^h}{1-\rho} - \sum\nolimits_{j=1}^{h-1} j\rho^j\end{aligned}$$

We notice that the variance of the simple moving average is related to the variance of the AR(1) process:

$$\operatorname{var}\left(\bar{X}_t^{(h)}\right) = \varphi(h)\operatorname{var}(X_t)$$

with:

$$\varphi(h) = \frac{1}{h}\left(1 + 2\rho\frac{1-\rho^{h-1}}{1-\rho} - 2\sum_{j=1}^{h-1}\frac{j}{h}\rho^j\right)$$

We give here the first values of $\varphi(h)$ when h varies between 1 and 5:

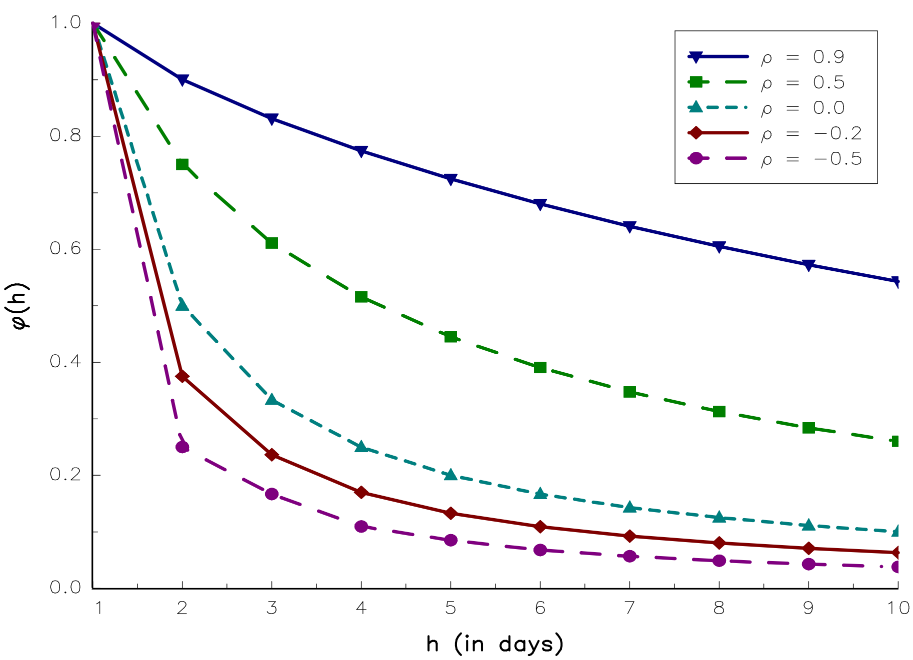
$$\begin{aligned}
\varphi(1) &= 1 \\
\varphi(2) &= \frac{1}{2} + \frac{1}{2}\rho \\
\varphi(3) &= \frac{1}{2} + \frac{4}{9}\rho + \frac{2}{9}\rho^2 \\
\varphi(4) &= \frac{1}{4} + \frac{3}{8}\rho + \frac{1}{4}\rho^2 + \frac{1}{8}\rho^3 \\
\varphi(5) &= \frac{1}{5} + \frac{8}{25}\rho + \frac{6}{25}\rho^2 + \frac{4}{25}\rho^3 + \frac{2}{25}\rho^4
\end{aligned}$$

More generally, we obtain the following recurrence for calculating $\varphi(h)$:

$$\varphi(h+1) = \left(\frac{h}{h+1}\right)^2\varphi(h) + \frac{1+\rho-2\rho^{h-1}}{(h+1)^2(1-\rho)}$$

In Figure 56, we report the value taken by $\varphi(h)$ for several levels of the parameter ρ.

Figure 56: Time-aggregation of the AR(1) process



C.6 The skewness coefficient

Let us consider a standardized random variable X with a probability density $f(x)$. The skewness coefficient γ_1 is defined as follows:

$$\gamma_1 = \int_{-\infty}^{+\infty} x^3 f(x)\,\mathrm{d}x$$

Another expression of γ_1 is:

$$\gamma_1 = \int_{0}^{+\infty} x^3 \left(f(x) - f(-x)\right)\,\mathrm{d}x$$

Lempérière *et al.* (2014a) propose an alternative measure of skewness. They consider the ranked P&L function $\Pi(p)$, whose expression is:

$$\Pi(p) = \int_{0}^{\mathbf{G}^{-1}(p)} x\left(f(x) - f(-x)\right)\,\mathrm{d}x$$

where $\mathbf{G}^{-1}(p)$ is the p-quantile of $|X|$. $\Pi(p)$ can be interpreted as the difference between the average amplitude of large negative and large positive returns[68]:

$$\Pi(p) = \mathbb{E}\left[\mathbf{1}\left\{X < -\mathbf{G}^{-1}(p)\right\}\cdot|X|\right] - \mathbb{E}\left[\mathbf{1}\left\{X > \mathbf{G}^{-1}(p)\right\}\cdot|X|\right]$$

They then define the skewness coefficient $\gamma_1^\star$ as follows:

$$\begin{aligned}
\gamma_1^\star &= -100\int_0^1 \Pi(p)\,\mathrm{d}p \\
&= -100\int_0^\infty \left(f(x) + f(-x)\right)\mathrm{d}x \int_0^x x\left(f(x) - f(-x)\right)\mathrm{d}x
\end{aligned}$$

C.7 Skewness aggregation

Let X and Y be two random variables. In what follows, we study the skewness of $X+Y$:

$$\gamma_1(X+Y) = \frac{\mu_3(X+Y)}{\mu_2^{3/2}(X+Y)}$$

where $\mu_n(X)$ is the n^{th} central moment of X. We remind that:

$$\mu_2(X+Y) = \mu_2(X) + \mu_2(Y) + 2\operatorname{cov}(X,Y)$$

and:

$$\mu_3(X+Y) = \mu_3(X) + \mu_3(Y) + 3\left(\operatorname{cov}(X,X,Y) + \operatorname{cov}(X,Y,Y)\right)$$

where:

$$\operatorname{cov}(X,Y,Z) = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])(Z - \mathbb{E}[Z])\right]$$

[68] We have:

$$\mathbb{E}[X] = \int_{-\infty}^{-\mathbf{G}^{-1}(p)} xf(x)\,\mathrm{d}x + \int_{-\mathbf{G}^{-1}(p)}^{\mathbf{G}^{-1}(p)} xf(x)\,\mathrm{d}x + \int_{\mathbf{G}^{-1}(p)}^{\infty} xf(x)\,\mathrm{d}x = 0$$

We deduce that:

$$\mathbb{E}\left[\mathbf{1}\left\{X < -\mathbf{G}^{-1}(p)\right\}\cdot X\right] + \Pi(p) + \mathbb{E}\left[\mathbf{1}\left\{X > \mathbf{G}^{-1}(p)\right\}\cdot X\right] = 0$$

The result is immediate.

C.7.1 Independent random variables

In this case, we obtain:

$$\mu_2(X+Y) = \sigma^2(X) + \sigma^2(Y)$$

and:

$$\mu_3(X+Y) = \mu_3(X) + \mu_3(Y)$$

We deduce that:

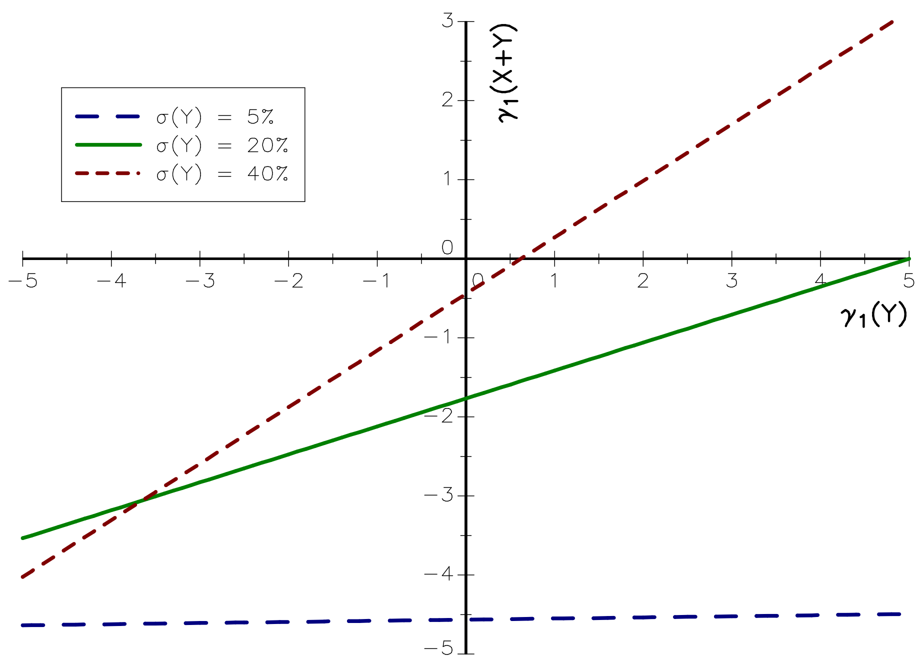
$$\begin{aligned} \gamma_1(X+Y) &= \frac{\mu_3(X)+\mu_3(Y)}{\left(\sigma^2(X)+\sigma^2(Y)\right)^{3/2}} \\ &= \gamma_1(X)\frac{\sigma^3(X)}{\left(\sigma^2(X)+\sigma^2(Y)\right)^{3/2}} + \gamma_1(Y)\frac{\sigma^3(Y)}{\left(\sigma^2(X)+\sigma^2(Y)\right)^{3/2}} \end{aligned} \tag{20}$$

It follows that if $\sigma(X) \gg \sigma(Y)$, the skewness of the sum $X+Y$ is close to the skewness of X:

$$\gamma_1(X+Y) \approx \gamma_1(X)$$

In order to illustrate this property, we report the skewness coefficient of $X+Y$ in Figure 57 when $\gamma_1(X) = -5$ and $\sigma(X) = 20\%$. This shows that diversification of skewness is difficult to achieve even if the random variables are independent.

Figure 57: Skewness coefficient of $X+Y$ when X and Y are independent



Remark 9 *If the random variables are i.i.d., the skewness coefficient of the sum is lower than the sum of skewness coefficients:*

$$\gamma_1(X_1+X_2) = \frac{\gamma_1(X)}{\sqrt{2}}$$

More generally, if $X_1, \ldots, X_n$ are i.i.d., we have:

$$\gamma_1 \left(\sum_{i=1}^{n} X_i \right) = \frac{\gamma_1 (X)}{\sqrt{n}}$$

C.7.2 Dependent random variables

In the general case, the expression of the skewness is:

$$\begin{aligned} \gamma_1 (X+Y) &= \gamma_1 (X) \frac{\sigma^3 (X)}{\sigma^{3/2} (X+Y)} + \gamma_1 (Y) \frac{\sigma^3 (Y)}{\sigma^{3/2} (X+Y)} + \\ & \quad \frac{3 \left(\operatorname{cov} (X, X, Y) + \operatorname{cov} (X, Y, Y) \right)}{\sigma^{3/2} (X+Y)} \end{aligned}$$

Using the definition of the coskewness:

$$\gamma_1 (X, Y, Z) = \frac{\operatorname{cov} (X, Y, Z)}{\sigma (X) \sigma (Y) \sigma (Z)}$$

we finally obtain:

$$\begin{aligned} \gamma_1 (X+Y) &= \gamma_1 (X) \frac{\sigma^3 (X)}{\sigma^{3/2} (X+Y)} + \gamma_1 (Y) \frac{\sigma^3 (Y)}{\sigma^{3/2} (X+Y)} + \\ & \quad \gamma_1 (X, X, Y) \frac{3\sigma^2 (X) \sigma (Y)}{\sigma^{3/2} (X+Y)} + \gamma_1 (X, Y, Y) \frac{3\sigma (X) \sigma^2 (Y)}{\sigma^{3/2} (X+Y)} \end{aligned} \tag{21}$$

The skewness of the sum is then a weighted average of skewness and coskewness coefficients.

If we assume that the coskewness coefficients are equal to zero, Formula (21) reduces to Formula (20), which was obtained in the independent case. This result holds even if the correlation between X and Y is high. For instance, this is the case for a correlated Gaussian vector. More generally, we have:

$$\begin{aligned} \gamma_1 (X, X, Y) &= \frac{\operatorname{cov} (X, X, Y)}{\sigma (X)^2 \sigma (Y)} \\ &= \frac{\operatorname{cov} \left(X^2, Y \right) - 2 \operatorname{cov} (X, Y) \mathbb{E} [X]}{\sigma (X)^2 \sigma (Y)} \end{aligned}$$

If $\mathbb{E} [X] = \mathbb{E} [Y] = 0$, we deduce that:

$$\gamma_1 (X, X, Y) = \frac{\rho \left(X^2, Y \right) \sigma \left(X^2 \right)}{\sigma (X)^2}$$

This shows that coskewness coefficients are sensitive to the correlations $\rho \left(X^2, Y \right)$ and $\rho \left(X, Y^2 \right)$.

C.7.3 An illustration with the log-normal case

We assume that (X, Y) follows a bivariate log-normal distribution. This implies that $\ln X \sim \mathcal{N} (\mu_X, \sigma_X)$ and $\ln Y \sim \mathcal{N} (\mu_Y, \sigma_Y)$. Moreover, we note ρ the correlation between $\ln X$ and $\ln Y$. By using the following result:

$$\mathbb{E} \left[e^{mX} \right] = \exp \left(m \mu_X + m^2 \frac{\sigma_X^2}{2} \right)$$

for $m \geq 1$, we obtain:

$$\mu_2(X) = \exp\left(2\mu_X + \sigma_X^2\right) \times \left(\exp\left(\sigma_X^2\right) - 1\right)$$

and:

$$\mu_3(X) == \left(\exp\left(3\sigma_X^2\right) - 3\exp\left(\sigma_X^2\right) + 2\right) \times \exp\left(3\mu_X + 3\frac{\sigma_X^2}{2}\right)$$

We deduce that the skewness of X is equal to:

$$\gamma_1(X) = \frac{\exp\left(3\sigma_X^2\right) - 3\exp\left(\sigma_X^2\right) + 2}{\left(\exp\left(\sigma_X^2\right) - 1\right)^{3/2}}$$

In order to find the skewness of the sum $X + Y$, we need a preliminary result. By denoting $Z = m \ln X + p \ln Y$, we have:

$$\mathbb{E}\left[e^Z\right] = \exp\left(\mu_Z + \frac{\sigma_Z^2}{2}\right)$$

where:

$$\mu_Z = m\mu_X + p\mu_Y$$

and:

$$\sigma_Z^2 = m^2\sigma_X^2 + p^2\sigma_Y^2 + 2mp\rho\sigma_X\sigma_Y$$

It follows that:

$$\mathbb{E}\left[X^m Y^p\right] = \exp\left(m\mu_X + p\mu_Y + \frac{m^2\sigma_X^2 + p^2\sigma_Y^2 + 2mp\rho_{X,Y}\sigma_X\sigma_Y}{2}\right)$$

Using the previous result, we show that the correlation between X and Y is equal to:

$$\rho_{X,Y} = \frac{\exp\left(\rho\sigma_X\sigma_Y\right) - 1}{\sqrt{\exp\left(\sigma_X^2\right) - 1}\sqrt{\exp\left(\sigma_Y^2\right) - 1}}$$

The variance of the sum is then:

$$\mu_2(X + Y) = \sigma^2(X) + \sigma^2(Y) + 2\rho_{X,Y}\sigma(X)\sigma(Y) \tag{22}$$

For the third moment of $X + Y$, we use the following formula:

$$\mu_3(X + Y) = \mu_3(X) + \mu_3(Y) + 3\left(\operatorname{cov}(X, X, Y) + \operatorname{cov}(X, Y, Y)\right) \tag{23}$$

where:

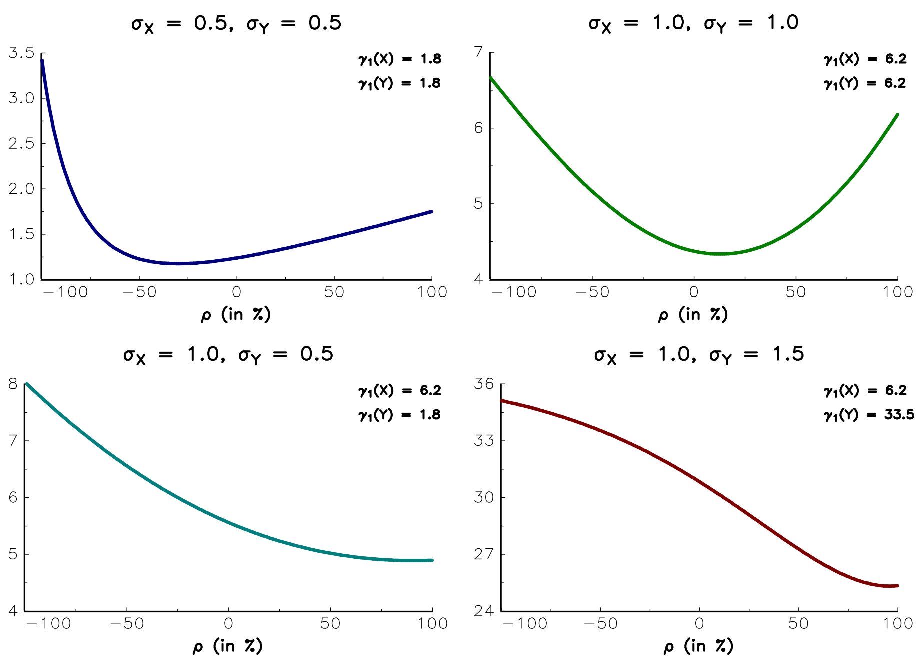
$$\begin{aligned}\operatorname{cov}(X, X, Y) \quad = \quad & \exp\left(2\mu_X + \sigma_X^2 + \mu_Y + \frac{\sigma_Y^2}{2}\right) \times \left(\exp\left(\rho\sigma_X\sigma_Y\right) - 1\right) \times \\ & \left(\exp\left(\sigma_X^2 + \rho\sigma_X\sigma_Y\right) + \exp\left(\sigma_X^2\right) - 2\right)\end{aligned}$$

The skewness of the sum $X + Y$ is then the ratio between (23) and (22).

In Figure 58, we report the skewness of $X + Y$ by assuming that $\mu_X = \mu_Y = 0.5$. For each panel, we consider a set of parameters (σ_X, σ_Y) and calculate $\gamma_1(X + Y)$ for

$\rho \in [-1, 1]$. When we consider the volatility risk measure, we know that the relationship with respect to the correlation parameter ρ is a monotone increasing function. With the skewness risk measure, the relationship is no longer monotone. For instance, in the first panel, we notice that the skewness of $(X + Y)$ firstly decreases, and then increases with respect to the parameter ρ. We also observe that the function depends on the values taken by $\gamma_1(X)$ and $\gamma_1(Y)$ (see the difference between first and second panels). When one skewness coefficient dominates the other (third and fourth panels), we obtain a decreasing function. Contrary to the volatility risk measure, a negative value of ρ implies a larger skewness than a positive value of ρ. This simple illustration shows that skewness aggregation is a difficult task when random variables are correlated.

Figure 58: Skewness coefficient of $X + Y$ when the random vector (X, Y) is log-normal



D Additional results

Table 14: Statistics of traditional risk premia

Ticker	Name	SR	$\mathcal{DD}^{\star}$	γ_1	$\gamma_1^{\star}$	$\mathcal{R}_{\sigma}$	γ_2
M2WD	MSCI ACWI index (USD)	0.10	5.02	-0.21	-1.13	1.13	7.98
M2WO	MSCI WORLD index (USD)	0.10	4.89	-0.20	-1.04	1.13	7.90
M2EF	MSCI EM index (USD)	0.18	4.93	-0.31	-1.33	1.18	7.92
M2US	MSCI United States index (USD)	0.11	3.71	-0.01	-0.93	1.07	8.91
M8EU	MSCI Europe index (EUR)	0.05	4.01	0.03	-0.85	1.05	5.67
M8JP	MSCI Japan index (JPY)	0.04	3.97	-0.24	-0.62	1.13	7.55
M1AP	MSCI AC Asia Pacific index (USD)	0.15	4.55	-0.29	-0.82	1.14	5.31
LGAGTRUH	Barclays Global Agg Govt index (USD)	1.15	2.97	-0.21	-0.81	1.08	1.37
LUAGTRUU	Barclays US Agg Govt index (USD)	0.68	2.17	-0.13	-0.85	1.06	2.03
LEATTREU	Barclays Euro Treasury Bond index (EUR)	0.88	2.73	-0.06	-0.70	1.05	3.43
SBJYL	Citi Japanese Govt Bond index (JPY)	0.81	3.15	-0.10	-0.30	1.07	5.72
LGCPTRUH	Barclays Global Agg Corporate index (USD)	1.00	5.47	-0.39	-0.96	1.17	1.80
LUACTRUU	Barclays US Agg Corporate index (USD)	0.75	4.77	-0.24	-0.99	1.10	2.00
LP05TREH	Barclays Pan European Agg Corporate index (EUR)	0.94	5.61	-0.54	-1.29	1.25	1.75
ADXY	Bloomberg JP Morgan Asia Dollar index (USD)	-0.57	5.40	-0.16	-0.75	1.08	7.19
DXY	Bloomberg Dollar Spot index (USD)	-0.26	7.40	-0.02	0.00	1.00	1.34
BCOMTR	Bloomberg Commodity index (USD)	-0.07	6.17	-0.18	-0.48	1.09	2.51

Table 15: Statistics of ARP generic indices

i	Name	SR	$\hat{\sigma}$ (SR)	$\mathcal{DD}^{\star}$	γ_1	$\gamma_1^{\star}$	$\mathcal{R}_{\sigma}$	γ_2	ρ	β
1	equities/carry/HDY/Global (USD)	0.20	0.15	5.30	-0.18	-0.70	1.12	10.78	0.95	0.99
2	equities/carry/HDY/DM (USD)	0.38	0.10	5.53	-0.29	-0.86	1.14	8.54	0.94	0.96
3	equities/carry/HDY/EM (USD)	0.41	0.27	5.06	-0.49	-1.33	1.22	4.57	0.96	0.85
4	equities/carry/HDY/US (USD)	0.34	0.09	4.04	-0.07	-0.55	1.06	10.98	0.94	0.92
5	equities/carry/HDY/Europe (EUR)	0.30	0.18	4.80	-0.03	-0.91	1.09	7.12	0.94	0.90
6	equities/carry/HDY/Asia Pacific (USD)	0.58	0.10	5.49	-0.71	-1.14	1.26	8.56	0.87	0.79
7	equities/event/merger arbitrage/DM (USD)	0.71	0.21	3.71	0.82	-0.40	1.06	49.82	0.45	0.21
8	equities/event/merger arbitrage/US (USD)	0.68	0.09	3.94	0.26	0.00	0.99	9.79	-0.01	-0.01

Continued on next page

Continued from previous page

i	Name	SR	$\hat{\sigma}$ (SR)	$\mathcal{DD}^{\star}$	γ_1	$\gamma_1^{\star}$	$\mathcal{R}_{\sigma}$	γ_2	ρ	β
9	equities/event/merger arbitrage/long/US (USD)	0.48	0.09	2.64	0.12	-0.96	1.11	16.39	0.83	0.63
10	equities/growth/US (USD)	0.14	0.09	3.45	-0.02	-1.03	1.06	6.33	0.97	1.10
11	equities/growth/Europe (EUR)	0.12	0.13	4.07	-0.01	-1.15	1.07	4.40	0.94	0.99
12	equities/growth/Japan (JPY)	0.05	0.10	4.03	-0.12	-0.28	1.08	6.34	0.92	0.97
13	equities/low volatility/EM (USD)	0.48	0.19	5.17	-0.47	-1.15	1.21	5.22	0.95	0.69
14	equities/low volatility/US (USD)	0.48	0.11	3.95	-0.02	-0.79	1.08	9.60	0.95	0.75
15	equities/low volatility/Europe (EUR)	0.45	0.08	5.14	-0.16	-1.07	1.14	7.80	0.96	0.72
16	equities/low volatility/Japan (JPY)	0.39	0.01	4.49	-0.85	-0.75	1.25	8.96	0.83	0.58
17	equities/low volatility/Asia Pacific (USD)	0.54	0.01	4.65	-0.54	-0.99	1.22	5.64	0.86	0.56
18	equities/momentum/cross-section/US (USD)	0.09	0.11	3.61	-0.11	-1.34	1.10	5.78	0.93	0.99
19	equities/momentum/cross-section/Europe (EUR)	0.41	0.08	4.27	-0.23	-1.41	1.16	4.21	0.90	0.88
20	equities/momentum/cross-section/Japan (JPY)	0.17	0.08	4.51	-0.56	-1.08	1.24	4.54	0.87	0.89
21	equities/quality/DM (USD)	0.53	0.00	4.12	0.08	-0.89	1.12	15.12	0.93	0.80
22	equities/quality/US (USD)	0.27	0.14	3.36	0.08	-0.83	1.06	9.67	0.98	0.95
23	equities/quality/Europe (EUR)	0.26	0.10	4.59	-0.07	-1.18	1.11	6.64	0.96	1.00
24	equities/quality/Asia Pacific (USD)	0.36	0.72	3.42	-0.04	-0.45	1.08	8.75	0.92	0.91
25	equities/reversal/time-series/US (USD)	0.54	0.15	1.68	3.06	1.27	0.72	72.66	0.15	0.09
26	equities/reversal/time-series/Europe (EUR)	0.52	0.03	1.46	1.86	0.97	0.81	43.70	0.24	0.09
27	equities/value/Global (USD)	0.45	0.20	4.77	-0.19	-0.97	1.12	8.36	0.97	1.10
28	equities/value/DM (USD)	0.35	0.02	4.64	-0.12	-0.90	1.09	8.09	0.93	1.23
29	equities/value/EM (USD)	-0.35	0.36	2.93	-0.34	-0.33	1.11	4.01	0.99	0.99
30	equities/value/US (USD)	0.34	0.12	3.88	-0.16	-0.74	1.10	9.53	0.97	1.06
31	equities/value/Europe (EUR)	0.16	0.12	4.39	0.03	-0.87	1.06	6.65	0.96	1.08
32	equities/value/Japan (JPY)	0.29	0.08	3.85	-0.04	-0.29	1.07	10.41	0.92	0.97
33	equities/value/Asia Pacific (USD)	0.45	0.17	4.89	-0.21	-1.22	1.14	9.17	0.83	0.90
34	equities/volatility/carry/US (USD)	0.64	0.19	12.74	-7.44	-4.39	3.82	112.06	-0.64	-0.06
35	equities/volatility/carry/Europe (EUR)	0.90	0.31	3.94	-1.96	-3.27	1.95	16.51	-0.69	-0.05
36	equities/volatility/term structure/US (USD)	0.78	0.01	4.23	0.16	0.63	0.93	7.98	0.21	0.02
37	rates/carry/FRB/US (USD)	1.04	0.01	2.15	-0.15	-0.18	1.04	3.74	0.46	0.57
38	rates/carry/FRB/Europe (EUR)	0.99	0.00	5.41	-0.41	-0.58	1.16	3.37	0.30	0.43

Continued on next page

Continued from previous page

i	Name	SR	$\hat{\sigma}$ (SR)	$\mathcal{DD}^{\star}$	γ_1	$\gamma_1^{\star}$	$\mathcal{R}_{\sigma}$	γ_2	ρ	β
39	rates/carry/TSS/DM (USD)	0.58	0.13	1.89	-0.17	-0.36	1.06	2.22	0.32	0.56
40	rates/momentum/time-series/DM (USD)	1.21	0.46	2.11	-0.27	0.44	1.06	5.83	0.49	0.78
41	rates/momentum/time-series/US (USD)	1.22	0.13	3.07	-0.18	0.45	1.07	5.61	0.59	0.68
42	rates/momentum/time-series/Europe (EUR)	0.91	0.28	2.64	-0.02	0.07	1.02	7.21	0.19	0.26
43	rates/volatility/carry/DM (USD)	1.57	0.15	3.38	-2.36	-1.16	1.56	27.55	0.06	0.13
44	rates/volatility/carry/US (USD)	1.10	0.16	2.52	-4.61	-0.78	1.90	75.17	0.00	0.01
45	rates/volatility/carry/Europe (EUR)	1.30	0.17	5.14	-0.50	-1.01	1.29	20.48	0.04	0.07
46	rates/volatility/carry/Japan (JPY)	0.84	0.14	6.51	-2.59	-1.04	1.59	31.28	0.21	0.67
47	currencies/carry/FRB/Global (USD)	0.89	0.06	4.45	-1.23	-1.38	1.32	16.89	-0.01	-0.01
48	currencies/carry/FRB/DM (USD)	0.34	0.07	5.60	-0.89	-1.62	1.31	9.65	-0.20	-0.21
49	currencies/carry/FRB/EM (USD)	0.96	0.28	5.24	-2.72	-1.45	1.44	60.47	0.44	0.77
50	currencies/momentum/time-series/Global (USD)	0.64	0.13	4.22	-0.31	-0.89	1.14	8.25	0.01	0.01
51	currencies/momentum/time-series/DM (USD)	0.44	0.08	3.51	0.04	-0.34	1.03	6.54	0.02	0.01
52	currencies/momentum/time-series/EM (USD)	1.19	0.40	2.11	-0.65	-0.79	1.15	11.57	-0.01	-0.02
53	currencies/value/economic model/DM (USD)	1.04	0.08	3.82	0.34	0.37	0.92	5.78	0.01	0.01
54	currencies/value/PPP/DM (USD)	0.66	0.02	4.90	-1.56	-0.32	1.22	28.30	0.36	0.17
55	commodities/carry/FRB/Global (USD)	0.90	0.24	2.45	-0.12	0.33	1.00	1.88	0.10	0.07
56	commodities/carry/TSS/Global (USD)	2.65	0.27	1.97	-0.79	0.37	1.08	14.74	-0.04	-0.01
57	commodities/liquidity/Global (USD)	2.62	0.04	1.14	-0.33	3.10	0.87	20.90	-0.28	-0.03
58	commodities/momentum/cross-section/Global (USD)	0.39	0.06	4.19	-0.19	-0.66	1.09	2.70	0.25	0.27
59	commodities/momentum/time-series/Global (USD)	0.52	0.30	4.63	-0.07	-0.41	1.04	3.85	0.17	0.09

Table 16: Out-of-sample performance ratio for HFR indices (2000-2015)

Strategy	TRP	ARP	SPX + ARP	TRP + ARP
HFRI	0.72	0.93	1.05	0.84
CTA	0.32	0.81	0.65	0.62
DS	0.71	0.73	0.76	0.81
ED	0.68	0.90	1.00	0.89
EH	0.78	1.04	0.95	0.97
EM	0.60	1.08	1.05	0.68
EMN	0.74	0.87	0.91	0.83
MA	0.69	0.85	0.90	0.91
MAC	0.71	1.22	1.21	1.27
RV	0.64	0.78	0.81	0.82
SB	1.97	1.76	1.39	1.69
FOF	0.86	1.07	1.04	0.93

Table 17: Out-of-sample performance ratio for EDHEC indices (2000-2015)

Strategy	TRP	ARP	SPX + ARP	TRP + ARP
CA	0.80	0.71	0.71	0.74
CTA	0.56	1.45	1.47	1.22
DS	0.54	0.78	0.80	0.66
ED	0.58	0.82	0.99	0.83
EM	0.67	1.13	1.00	0.61
EMN	0.70	0.84	0.83	0.84
FIA	0.79	0.77	0.78	0.88
GM	0.74	1.01	1.03	0.94
LSE	0.66	0.93	0.98	0.84
MA	0.58	0.83	0.85	0.81
RV	0.65	0.84	0.82	0.74
SB	1.94	1.80	1.44	1.52
FOF	0.84	0.96	1.01	0.91

Table 18: The 10 most frequent risk premia for HFR indices (2000-2008)

Strategy	Frequency	Risk premia
CTA	88.9	SPX
	71.3	equities/growth/US
	42.6	equities/event/merger arbitrage/DM
	40.7	equities/low volatility/Japan
	38.0	MXEF
	36.1	equities/momentum/cross-section/Europe
	34.3	currencies/momentum/time-series/EM
	32.4	equities/low volatility/EM
	28.7	commodities/momentum/cross-section
	28.7	equities/momentum/cross-section/US
EH	100.0	SPX
	75.0	equities/growth/US
	65.7	MXEF
	58.3	equities/event/merger arbitrage/DM
	51.9	RTY
	47.2	equities/volatility/carry/US
	44.4	equities/momentum/cross-section/US
	33.3	equities/value/DM
	33.3	equities/momentum/cross-section/Europe
	32.4	HY
MA	95.4	equities/event/merger arbitrage/DM
	51.9	SPX
	50.9	HY
	46.3	RTY
	45.4	equities/quality/Europe
	40.7	equities/reversal/time-series/US
	38.9	equities/volatility/carry/US
	34.3	equities/momentum/cross-section/Europe
	33.3	equities/growth/US
	28.7	equities/low volatility/EM
RV	78.7	equities/event/merger arbitrage/DM
	67.6	HY
	56.5	equities/momentum/cross-section/Europe
	51.9	currencies/carry/FRB/DM
	42.6	equities/volatility/carry/US
	41.7	currencies/carry/FRB/EM
	36.1	MXEF
	35.2	EMBI
	29.6	equities/growth/Europe
	28.7	equities/value/DM

Table 19: The 10 most frequent risk premia for HFR indices (2009-2015)

Strategy	Frequency	Risk premia
CTA	88.0	commodities/momentum/time-series
	77.1	currencies/momentum/time-series/DM
	65.1	commodities/momentum/cross-section
	61.4	rates/momentum/time-series/DM
	47.0	JPY
	47.0	equities/value/US
	36.1	currencies/momentum/time-series/EM
	34.9	equities/volatility/term structure/US
	25.3	equities/event/merger arbitrage/US
	24.1	currencies/value/economic model/DM
EH	100.0	SPX
	85.5	HY
	79.5	equities/growth/US
	72.3	equities/low volatility/Asia Pacific
	67.5	equities/low volatility/EM
	57.8	equities/low volatility/US
	55.4	GSCI
	54.2	equities/volatility/carry/US
	49.4	currencies/carry/FRB/EM
	43.4	currencies/carry/FRB/DM
MA	83.1	SPX
	78.3	equities/event/merger arbitrage/DM
	77.1	HY
	68.7	equities/volatility/carry/US
	49.4	equities/volatility/carry/Europe
	36.1	rates/volatility/carry/Europe
	30.1	equities/quality/DM
	30.1	equities/quality/Europe
	28.9	currencies/carry/FRB/DM
	25.3	commodities/liquidity
RV	100.0	HY
	100.0	equities/volatility/carry/US
	42.2	SPX
	33.7	equities/event/merger arbitrage/US
	32.5	currencies/carry/FRB/EM
	30.1	equities/value/DM
	28.9	equities/reversal/time-series/Europe
	28.9	JPY
	28.9	GSCI
	28.9	equities/volatility/term structure/US

Table 20: The 10 most frequent risk premia for EDHEC indices (2000-2008)

Strategy	Frequency	Risk premia
CTA	98.1	currencies/momentum/time-series/DM
	72.2	rates/momentum/time-series/DM
	64.8	commodities/momentum/time-series
	40.7	currencies/momentum/time-series/EM
	34.3	RTY
	34.3	commodities/momentum/cross-section
	33.3	commodities/carry/TSS
	32.4	GSCI
	29.6	rates/carry/FRB/Europe
	25.9	commodities/liquidity
LSE	98.1	SPX
	75.0	equities/growth/US
	58.3	equities/event/merger arbitrage/DM
	55.6	MXEF
	50.9	equities/volatility/carry/US
	49.1	equities/momentum/cross-section/US
	46.3	RTY
	44.4	currencies/carry/FRB/EM
	31.5	HY
	31.5	equities/value/Global
MA	98.1	equities/event/merger arbitrage/DM
	60.2	HY
	57.4	equities/quality/Europe
	55.6	SPX
	50.0	equities/growth/US
	43.5	equities/reversal/time-series/US
	40.7	equities/volatility/carry/US
	37.0	equities/momentum/cross-section/Europe
	26.9	EMBI
	25.9	currencies/carry/FRB/EM
RV	86.1	equities/event/merger arbitrage/DM
	67.6	HY
	62.0	SPX
	52.8	currencies/carry/FRB/DM
	48.1	equities/quality/Europe
	41.7	equities/volatility/carry/US
	39.8	MXEF
	37.0	equities/growth/US
	36.1	equities/growth/Europe
	33.3	equities/value/Global

Table 21: The 10 most frequent risk premia for EDHEC indices (2009-2015)

Strategy	Frequency	Risk premia
CTA	83.1	commodities/momentum/time-series
	73.5	rates/momentum/time-series/DM
	67.5	currencies/momentum/time-series/DM
	66.3	commodities/momentum/cross-section
	43.4	JPY
	41.0	equities/value/US
	37.3	GOLD
	36.1	currencies/momentum/time-series/EM
	36.1	equities/event/merger arbitrage/US
	36.1	equities/volatility/term structure/US
LSE	98.8	SPX
	83.1	HY
	81.9	equities/low volatility/Asia Pacific
	75.9	equities/growth/US
	65.1	equities/volatility/carry/US
	48.2	equities/low volatility/US
	39.8	currencies/carry/FRB/EM
	38.6	equities/low volatility/EM
	34.9	equities/volatility/term structure/US
	31.3	equities/quality/DM
MA	88.0	HY
	78.3	equities/event/merger arbitrage/DM
	66.3	equities/volatility/carry/US
	62.7	SPX
	62.7	equities/volatility/carry/Europe
	36.1	equities/quality/DM
	26.5	commodities/carry/TSS
	26.5	commodities/liquidity
	26.5	equities/event/merger arbitrage/US
	26.5	currencies/carry/FRB/DM
RV	98.8	equities/volatility/carry/US
	95.2	HY
	74.7	SPX
	39.8	GSCI
	33.7	equities/event/merger arbitrage/DM
	30.1	commodities/liquidity
	28.9	equities/value/DM
	28.9	currencies/carry/FRB/EM
	26.5	equities/event/merger arbitrage/US
	25.3	currencies/value/economic model/DM

Figure 59: Density function of empirical and Fama-MacBeth estimators ($T = 36$ months)

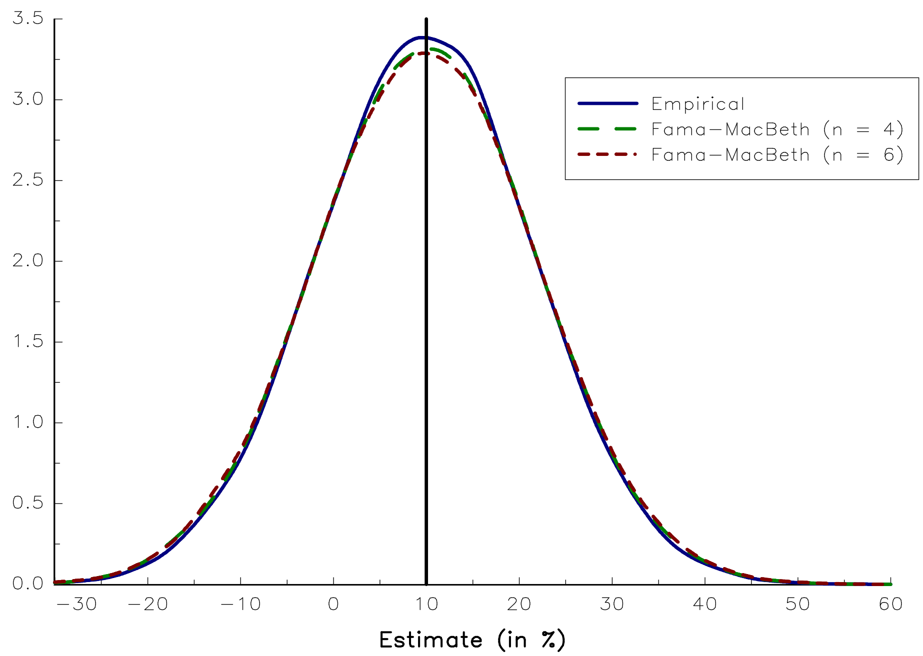


Figure 60: ARP backtests (credit)

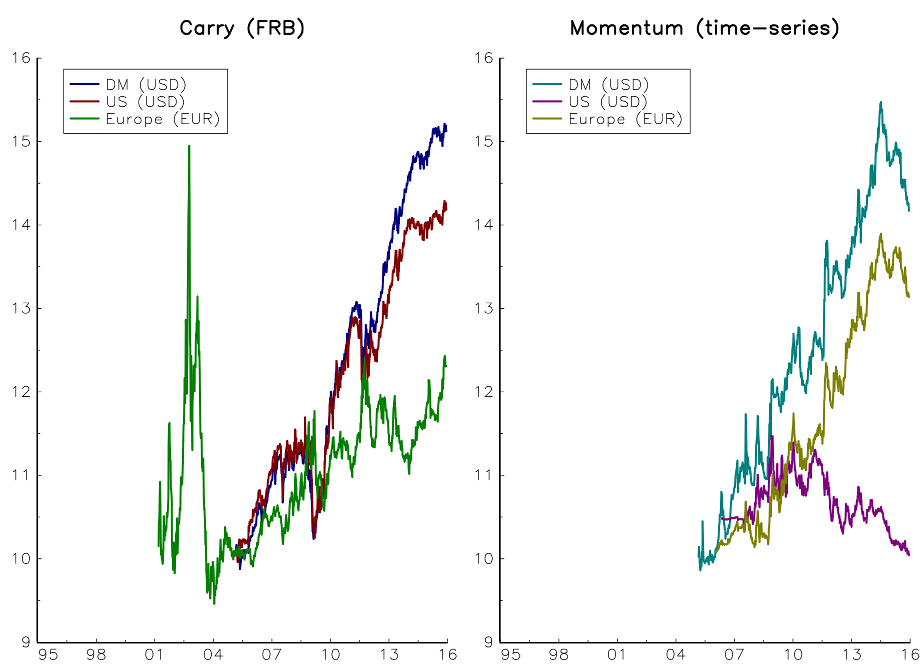


Figure 61: Payoff function of TRP wrt to equities

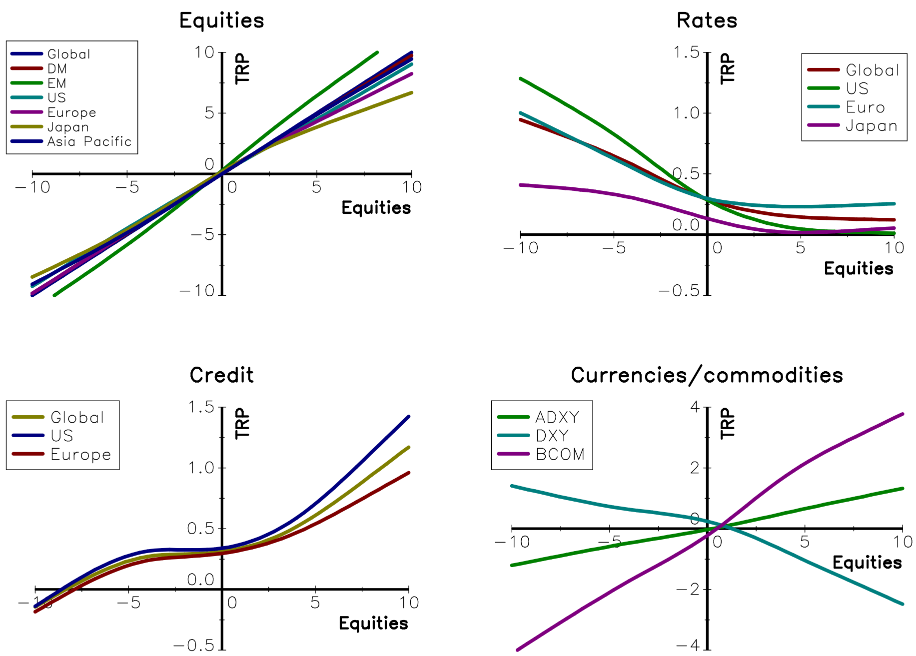


Figure 62: Payoff function of TRP wrt to rates

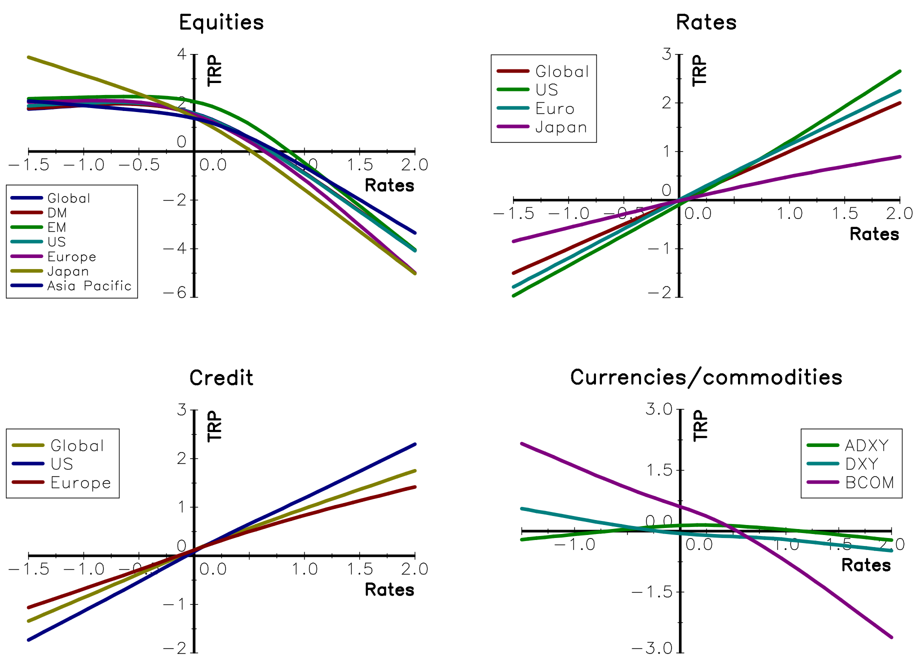


Figure 63: Payoff function of equities ARP wrt to equities

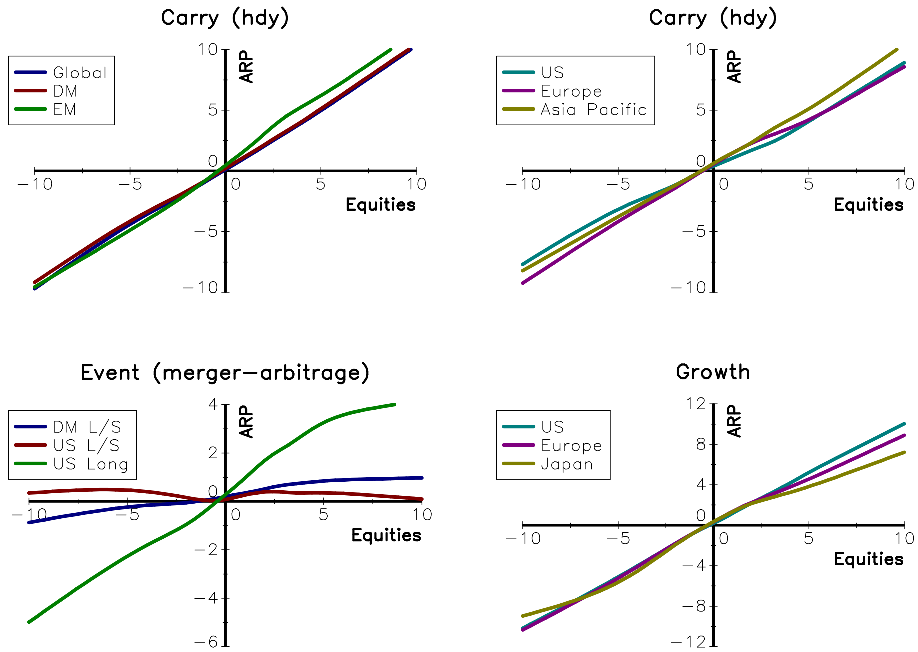


Figure 64: Payoff function of equities ARP wrt to equities

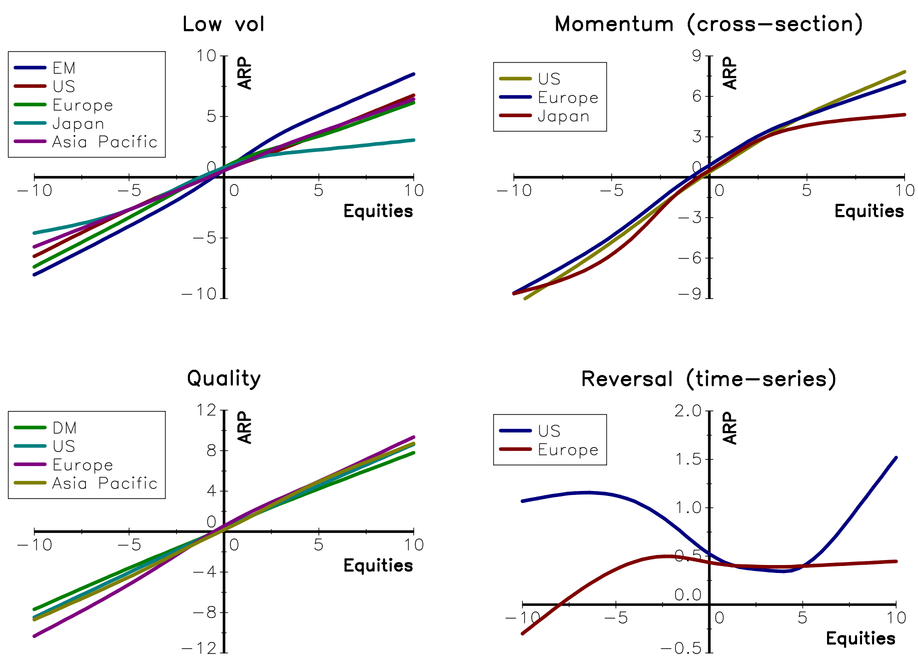


Figure 65: Payoff function of equities ARP wrt to equities

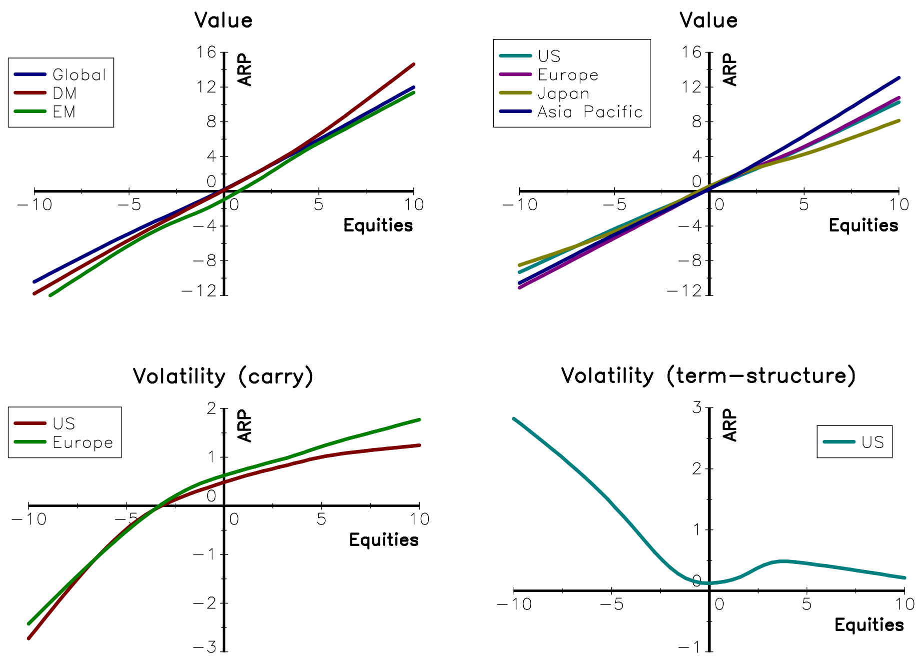


Figure 66: Payoff function of rates ARP wrt to equities

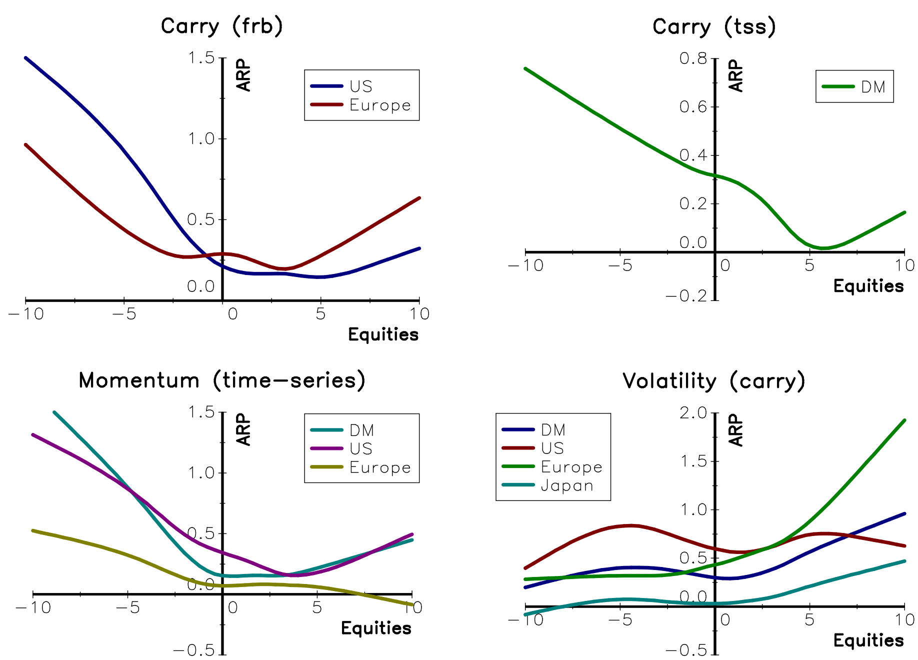


Figure 67: Payoff function of currencies ARP wrt to equities

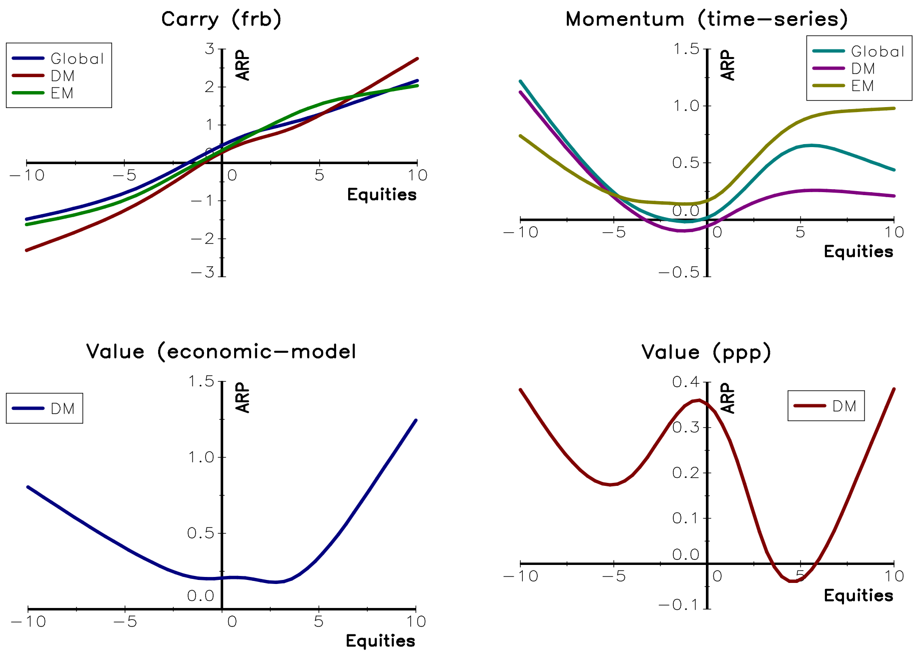


Figure 68: Payoff function of commodities ARP wrt to equities

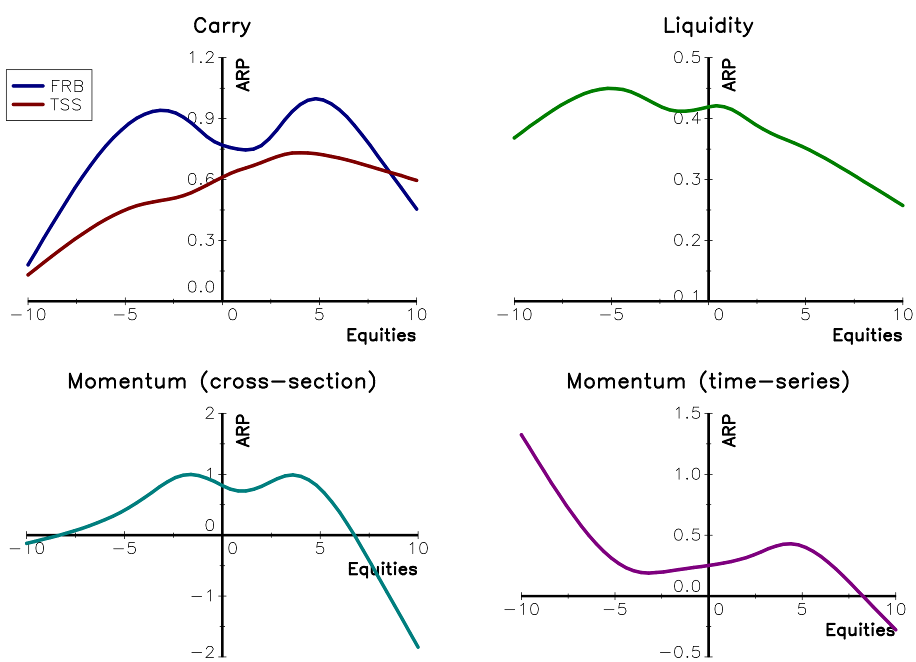


Figure 69: Payoff function of equities ARP wrt to rates

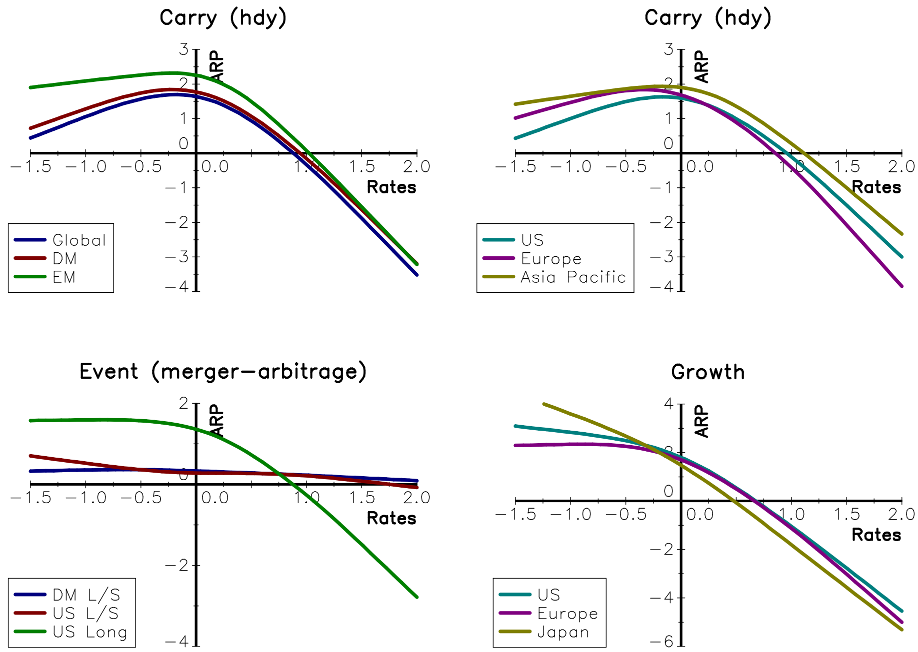


Figure 70: Payoff function of equities ARP wrt to rates

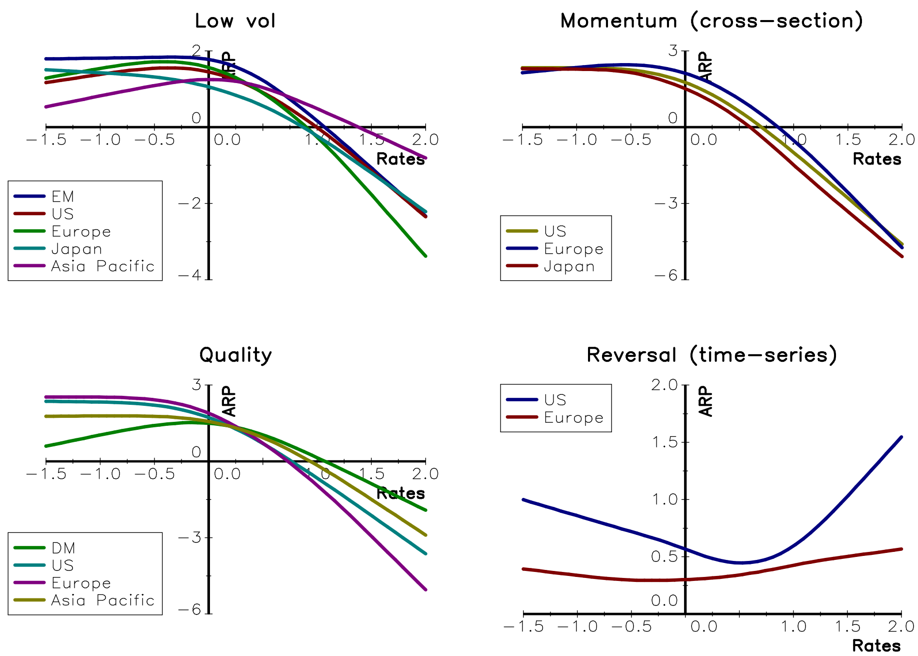


Figure 71: Payoff function of equities ARP wrt to rates

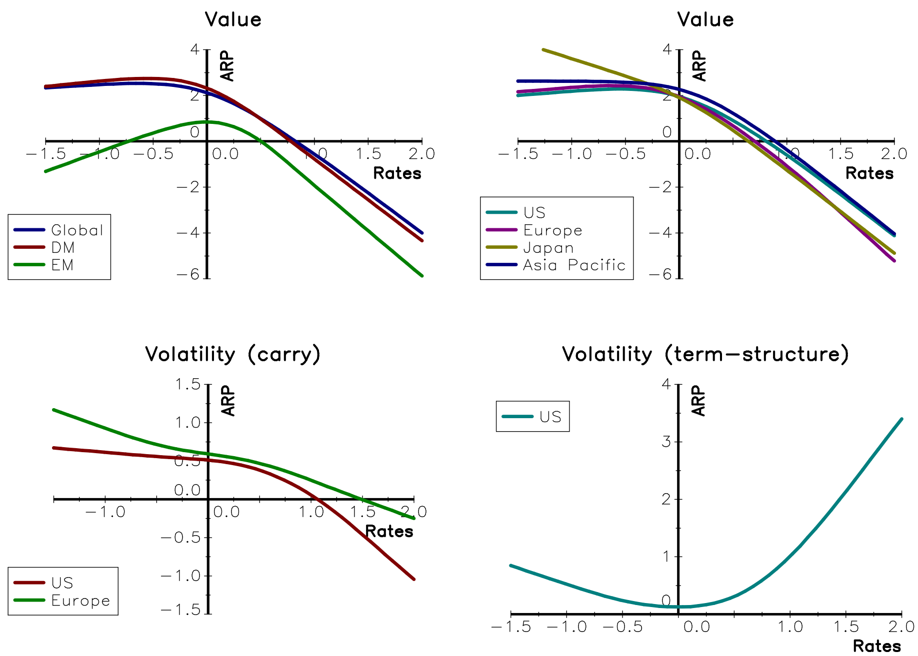


Figure 72: Payoff function of rates ARP wrt to rates

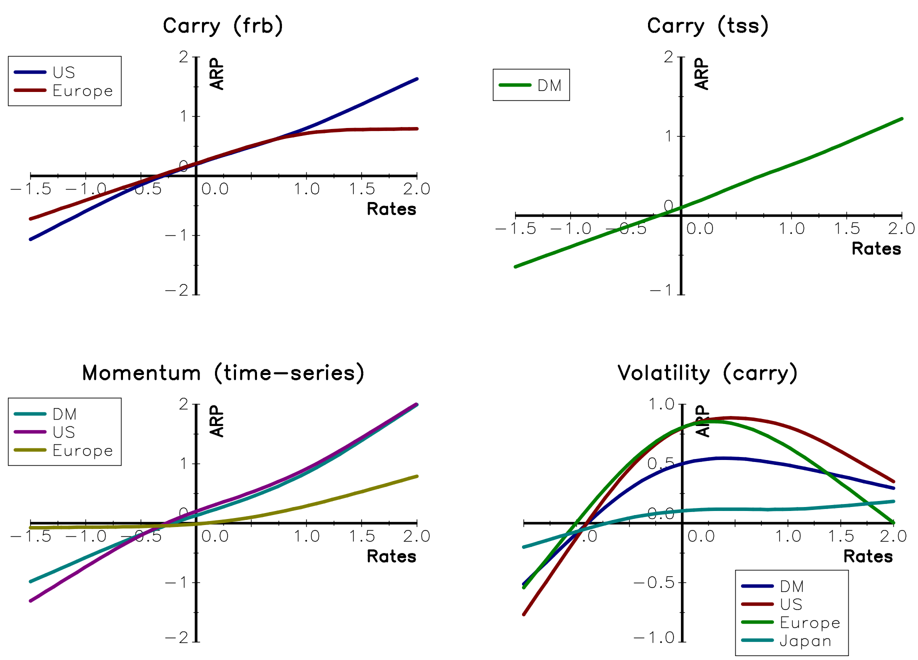


Figure 73: Payoff function of currencies ARP wrt to rates

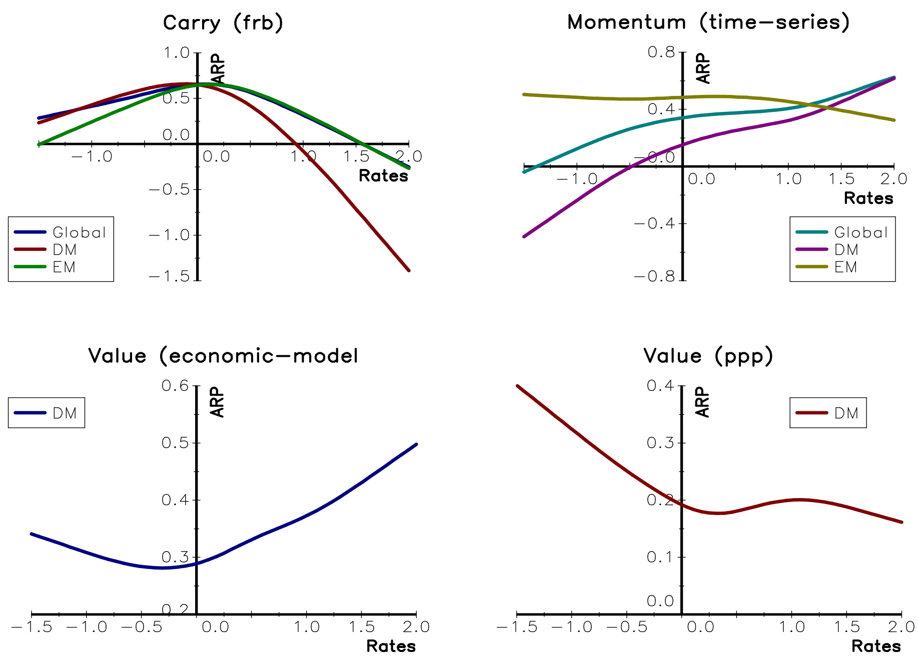


Figure 74: Payoff function of commodities ARP wrt to rates

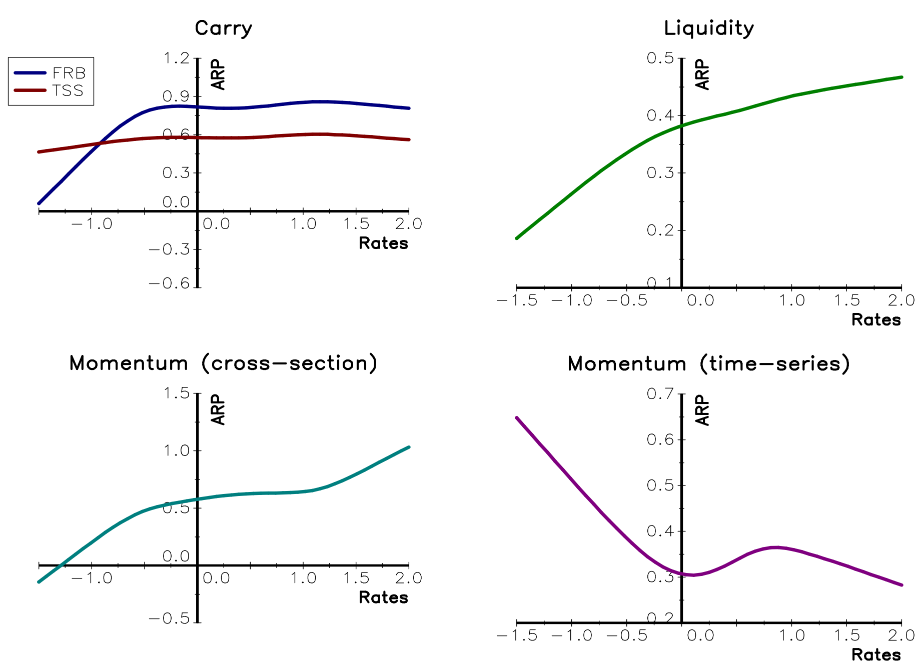


Figure 75: In-sample $\mathbf{R}^2$ (in %) with five factors for HF indices (2000-2015)

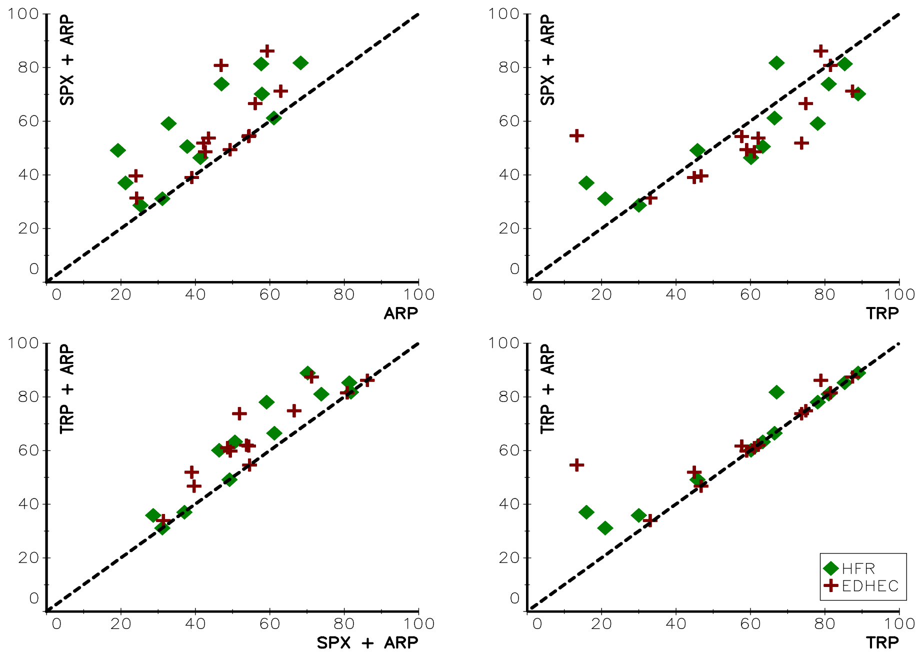


Figure 76: In-sample $\mathbf{R}^2$ (in %) with 10 factors for HF indices (2000-2015)

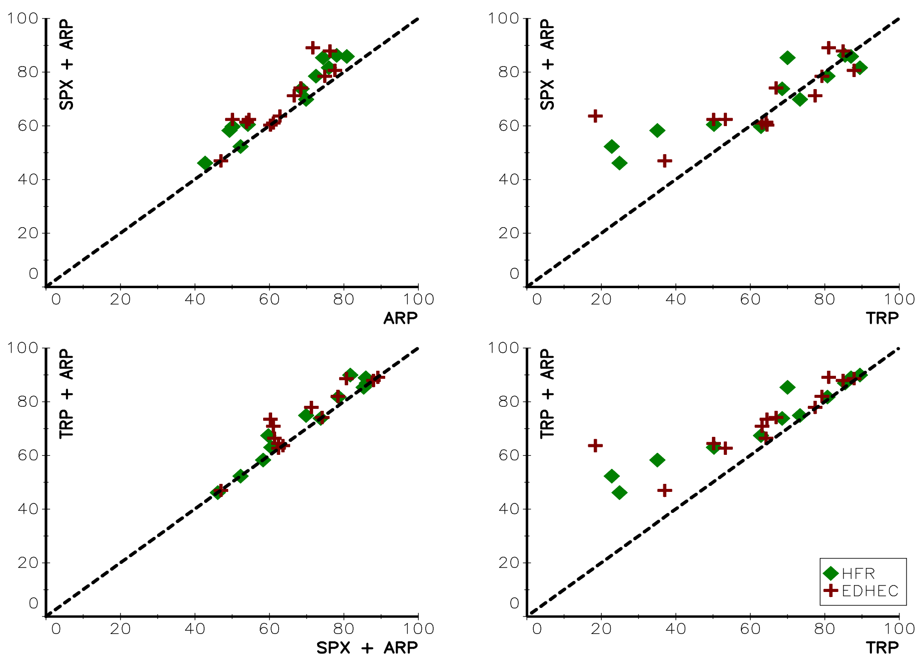


Figure 77: Yearly TRP/ARP frequencies for HFR indices (2000-2015)

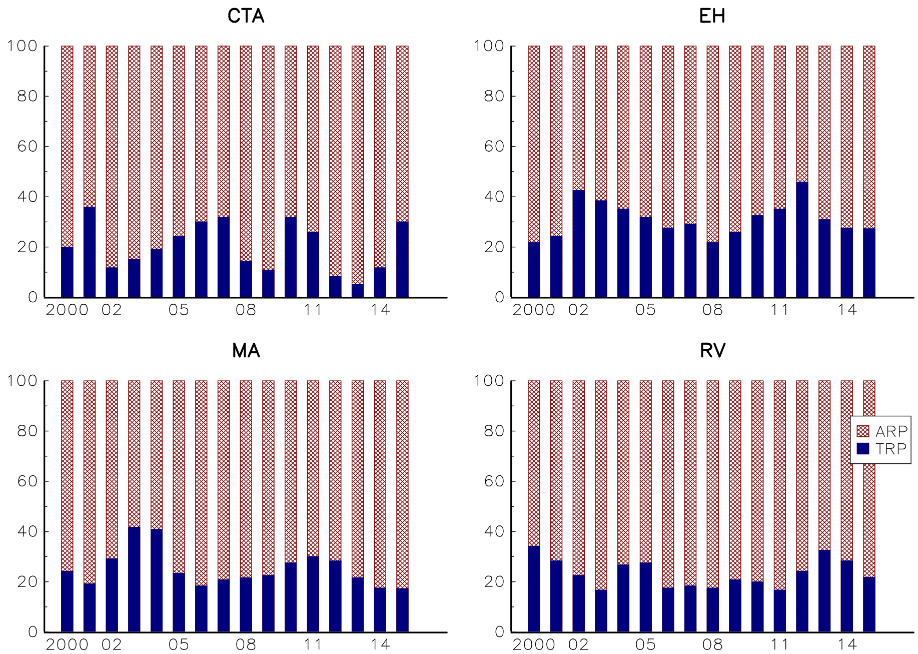


Figure 78: Yearly TRP/ARP frequencies for EDHEC indices (2000-2015)